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**İBN SÎNÂ**  
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## IBN SĪNĀ AS A MATHEMATICIAN

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Abū ‘Alī al-Husayn Ibn ‘Abdallah Ibn Sīnā (Avicenna) [980-1037] is known chiefly as a philosopher and physician. However, in his autobiography (preserved in Ibn Abī Uṣaybi‘a’s *‘Uyun al-abā’ fī ṭabaqāt al-aṭibbā’*), Ibn Sīnā makes numerous references to his early studies in mathematics, starting with lessons in the “Indian arithmetic”, and continuing through Euclid and the *Almagest*. He also maintained a scientific correspondence and rivalry with al-Bīrūnī, who is known, among other things, for contributions to mathematical geography and numerical approximation. Consequently, it is of interest to examine Ibn Sīnā’s own mathematical work, particularly in this millenary year of his birth. To date, no comprehensive study has been devoted to this subject. Limited accounts appear in a short thesis of Lokotsch (1912), which discusses Ibn Sīnā’s commentaries on the planimetric books of Euclid, and in the notes of Woepcke and Tannery on the “method of casting 9’s.” The present writers examine the extant treatises of Ibn Sīnā, in particular, *Kitāb al-Shifā’*, *Dānish Nāmā’i ‘Alā’i*, and *al-Najāt* to offer a comprehensive picture of his mathematical accomplishments, and thereby, to assess the validity of Ṣarton’s claim that “Ibn Sīnā’s interest in mathematics was philosophical rather than technical.”

## I. Mathematical Education

Abū ‘Alī al-Ḥusayn Ibn ‘Abdallāh Ibn Sīnā’ (Avicenna) [980-1037] is well known for his work in philosophy and medicine, but few studies detail his accomplishments in mathematics. Woepcke, Eneström, de Montferrier and Tannery<sup>2</sup>, for example, each devoted a short article to Ibn Sīnā’s arithmetic or to his account of the method of casting out nines, while Lokotsch, in a 27 page Bonn University Dissertation, *Avicenna als Matkematiker*, considered only the summary of the planimetric books of Euclid, appearing in Kitāb al-Shifā.<sup>3</sup> A further note on the contents page of this 1912 dissertation promises a German translation of the mathematical writings of Avicenna within two years, but World War I intervened, and while Lokotsch survived, the projected translation evidently never was printed.<sup>4</sup> Later writers usually pass over Ibn Sīnā’s mathematics with a flat statement of his divisions of mathematics in the Shifā’ into four principal (theoretical) sciences and four subordinate (practical) ones : arithmetic (‘ilm al-ḥisāb), geometry (‘ilm al-handasa), geography and astronomy (ilm al-hay’a), music (‘ilm al-mūsīqī), with respective practical subsciences: Hindu mathematics and algebra, mechanics, preparation of zījs and calendars, use of musical instruments.<sup>5</sup> Of course, compared to the brilliant work of his elder contemporary al-Bīrūnī (973-1048), with whom he competed, all these encyclopedic summaries appear flat and unexciting. Possibly this, or similar evidence, prompted Sarton to declare that “Ibn Sīnā’s interest in mathematics was philosophical rather than technical, ...

<sup>1</sup> Recent biographical accounts appear in the DSB articles of Anawati (9) and Iskandar (53). For other accounts, see S. H. Nasr (81), and his text *Science and Civilization in Islam* (82); also the biographies by B. Carra de Vaux (19) and S. M. Afnan (1).

<sup>2</sup> Woepcke (122), Eneström (32), de Montferrier (76). Cantor (18) covers the same ground.

<sup>3</sup> Lokotsch (67). The paper of Ahadova (4) essentially does this work over, using the geometric part of the Dānesh Nāmeḥ ‘Alā’ī instead of the Shifā’.

<sup>4</sup> Says Lokotsch, “Die deutsche Übersetzung der mathematischen Schriften Avicennas soll nebst sachlichen Anmerkungen in ungefähr zwei Jahren in Buchform erscheinen.” The *Jahrbuch über der Fortschritte der Mathematik*, which carries an entry for the dissertation, does not carry one for the promised translation, and a search by the Prussian State Library (Berlin) (1980) failed to turn up any record of one. A photocopy of the dissertation in the possession of Prof. Dr. F. Sezgin indicates that Lokotsch donated the dissertation to the Carl v. Kothschildsche Bibliothek in Frankfurt in 1929.

<sup>5</sup> See Anawati (9), for example.

[as to be expected] in a late Neoplatonist.”<sup>6</sup> While the present study may not overturn this statement, it will establish that it is a simplification, and it will offer the reader ample mathematical citations from Ibn Sīnā’s work. Let us add that for want of time, we shall restrict our discussion of Ibn Sīnā’s mathematics to mathematics in modern sense. Moreover, we shall call the reader’s attention to some excellent, and rather recent Russian studies on certain aspects of Ibn Sīnā’s mathematics.

Before listing and examining these mathematical works, let us first establish what is known of Ibn Sīnā’s mathematical education, and his general attitude toward mathematics. The principal source of this material is an autobiographical fragment, edited and supplemented by his student ‘Abd al-Wāḥid Ibn Muḥammad al-Jūzajānī (11th c), and known as *Sīrat al-Shaykh al-Ra’īs* (Life of Ibn Sīnā).<sup>7</sup> In the fragment, Ibn Sīnā recounts that he had completed his Qurānic education by the age of ten,<sup>8</sup> and that his father’s house in Bukhārā served as a center for discussions of “philosophy, geometry and Indian calculations.”<sup>9</sup> He adds that both he and his father studied the *Rasā’il Ikhwān al-Ṣafā’*, (*Epistles of the Ikhwān al-Ṣafā’*), an important point to consider, when we shortly examine the structure of Ibn Sīnā’s mathematical texts. His father then sent him to learn “Indian calculation” (*al-ḥisāb al-Hindī*) and algebra (*al-jabr wa’l-muqābāla*) from a vegetable seller known as *al-Maḥmūd al-Massāḥī*, and at the same time he read logic and geometry with the philosopher Abu ‘Abd Allāh al-Nātilī. Ibn Sīnā refers to him, in fact, as one who “claimed to know philosophy,”<sup>10</sup> and had no knowledge of the deeper parts of logic. Together, they started the *Elements* of Euclid and got through the “first five or six figures”<sup>11</sup>-possibly the first eight of Book I-when Ibn Sīnā got impatient and “undertook the solution of the rest of the book in its entirety”<sup>12</sup> by himself. He then began

<sup>6</sup> Sarton (94), I, 710.

<sup>7</sup> We refer to the edition and commentaries of Gohlman (38), in what follows.

<sup>8</sup> Tradition has it that Ibn al-‘Arabi and ‘Umar Ibn Aḥmad mastered the Qurān by age 9, while al-Ṭabarī seems to hold a record for mastery by age 7. See for example Totah, K. A., *Contribution of the Arabs to Education*, N. Y., 1926, p. 49, or his references : *Al-Maqqari*, vol. I, 343; Yaḳūt; *Dictionary of Learned Men*, VI, 30, 36-37.

<sup>9</sup> See Gohlman (38), pages 19, 21, 23, 25, 31 respectively.

<sup>10</sup> See Gohlman (38), pages 19, 21, 23, 25, 31 respectively.

<sup>11</sup> See Gohlman (38), pages 19, 21, 23, 25, 31 respectively.

<sup>12</sup> See Gohlman (38), pages 19, 21, 23, 25, 31 respectively.

Ptolemy's *Almagest*, but al-Nātilī begged off when Ibn Sīnā completed the first nine sections of Book I, saying, "Take over reading and solving them [the geometrical figures] by yourself; then show them to me, so that I can explain to you what is right and what is wrong."<sup>13</sup> Given al-Nātilī's performance with the *Elements*, his attitude toward the *Almagest* is understandable; section 10 of Book I assumes the reader has ready familiarity with Euclid. The very first figure is a simultaneous construction of a regular decagon and pentagon requiring, in order: *Elements* II, 6; I, 47; VI, definition 3; XIII, 9; IV, 15 corollary; XIII, 10, as justifications to be supplied. In fact, al-Nātilī was of little help, and Ibn Sīnā became the lecturer. Whether this continued through the whole of the *Almagest* is uncertain, for the text of *Sīrat al-Shaykh al-Ra'īs* simply reports that al-Nātilī left for Gurgānj, leaving Ibn Sīnā to his own devices. These, according to the fragment, consisted of readings in natural sciences ('ilm ilāhī), metaphysics (al-ilāhiyyāt), and medicine. By the age of 16, Ibn Sīnā claimed a mastery of medicine, as well as jurisprudence, and turned again to the serious study of philosophy for about a year and a half. In the text, he remarks, at this point, "Everything which I knew at that time is just as I know it now; I have not added anything to it to this day."<sup>14</sup> He then took up the study of Aristotle's *Metaphysics*, and claimed that he read it so often (40 times) without comprehension, until he acquired, by chance, al-Fārābī's commentary, possibly the *Kitāb al-ḥuruf*,<sup>15</sup> which made Aristotle's intention clear. At about this time, an invitation to share in the treatment of the ailing governor of Bukhāra (Nūḥ Ibn Manṣūr) gave him access to the governor's library, where he read in all the sciences. Unfortunately, the fragment gives no specific authors or titles. By this time Ibn Sīnā was 18; and the text reiterates his claim to have acquired no new knowledge since that time!

Thus, according to the *Sīrat*, Ibn Sīnā's mathematical training consisted of (possibly) the encyclopedia articles of the *Ikhwan al-Ṣafā'*, some algebra and Hindu arithmetic, Euclid's *Elements*, Ptolemy's *Syntaxis Mathematica* (*Almagest*), a classical compendium of applied mathematics, as well as of

<sup>13</sup> See Gohlman (38), pages 19, 21, 23, 25, 31 respectively.

<sup>14</sup> See Gohlman (38), pages 19, 21, 23, 25, 31 respectively.

<sup>15</sup> Gohlman (38), note 30, page 122.

astronomy, and (probably) the assimilation of much of Aristotle's notions upon the foundations of mathematics.

At this point al-Jūzajani continues the *Sīrat* on his own, by introducing himself as a student who read the *Almagest* and studied logic with Ibn Sīnā. He then offers a bibliography of 46 works of his teacher, 5 of which appear to contain mathematical studies: the *Shifā'*, *Najāt*, *Ala'ī*, Summary of Euclid, The Finite and Infinite. The order of composition is not particularly certain from al-Jūzajani's account, although he does mention that Ibn Sīnā had already written book one of the *Qānūn* before embarking on the *Shifā'*, and that he would often write 50 pages a day, even outlining whole books without consulting references. Al-Jūzajani, who is known for his own work on astronomy, *Kitāb Kaifiyat al-aflāk* which influenced the work of Quṭb al-Dīn al-Shīrāzī [1236-1311],<sup>16</sup> goes on to underline his master's competence in general with remarks such as: "he was not outclassed in any of the sciences," during the Friday night gatherings of the learned at the court of Amir 'Alā al-Dawla. As one would expect, al-Jūzajani offers no particular opinions on his master's mathematics, except to note that Ibn Sīnā "presented..., in the Arithmetic [of *Shifā'*] some excellent numerical properties,"<sup>17</sup> and to describe some additions he made to the drafts of the summaries of the *Elements* and the *Almagest* for the *Shifā'*. As will soon become apparent, al-Jūzajani does, however, offer valuable insight into Ibn Sīnā's motivation for writing these summaries, when he introduces sections of similar material in the *Dānesh Nāmeḥ*. Another early biographer, Ibn Funduq al-Bayhaqī (d. 1170), in the *Tatimmat ṣiwān al-ḥikma* (Supplement to the "Repository of Wisdom"), however, pronounces a judgment which has all elements of Sarton's. In fact, he says Ibn Sīnā was not much of a mathematician simply because "a person who has tasted the sweetness of metaphysics is niggardly in spending his thoughts in mathematics, so he fancies it from time to time and then abandons it".<sup>18</sup>

<sup>16</sup> Sezgin (98), vol. VI, 280-281. More information on al-Shīrāzī appears in S. H. Nasr's biographical sketch, DSB XI, 247-253. Al-Shīrāzī was, most notably, a pupil of Naṣīr al-Dīn al-Tūsī. He refers to al-Jūzajani's work while discussing his own model for the motion of Mercury.

<sup>17</sup> Gohlman (38), 65-66.

<sup>18</sup> Gohlman (38), 122, note 27. See also Ibn Funduq: *Tatimmat Ṣiwān al-ḥikma*. Muḥammad Shaffī, ed. Lahore, 1935, volume 1, p. 42. This supplement continues the *Ṣiwān al-ḥikma* of Abū Sulaymān al-Sijistāni (d. 985); so the mathematical opinion is Ibn Funduq's own. Apparently, he

Al-Jūzajānī then briefly takes up Ibn Sīnā's involvement in an observational astronomy project for al-Dawla,<sup>19</sup> and in a philological argument with Abū Mānṣūr al-Jabbān. The latter seems to have taken some three years to conclude-in Ibn Sīnā's favor-and is of interest as a comment on Ibn Sīnā's character. It also gives the lie to his constant claim of "learning nothing new from age 18." After al-Jūzajānī's account of his master's death, another (unknown) writer adds a bibliographic postscript. This lists 92 titles of Ibn Sīnā, includes the mathematical ones given by al-Jūzajānī, with some changes; for example, "Finite and Infinite" becomes "Infinity", and The Summary of Euclid is recorded as (possibly) "for the Najāt." It adds two further items: a letter to Abū Sahl al-Masīḥi on the Angle, and a "Summary [of the position] that the Angle which is formed by the Circumference and the Tangent has no magnitude." Mahdavi and Anawati both identify these treatises on the angle as being the same.<sup>20</sup>

## II. Mathematical Texts

Having briefly remarked on the mathematical texts of Ibn Sīnā as reported in *Sīrat al-Shaykh al-Ra'īs* and its anonymous appended bibliography, let us list them in detail, with manuscript locations, conjectural dates of composition, and evidence of translation into Latin in the Middle Ages. This will both define the limits of our main area of discussion, and, later, allow us to trace the influence of Ibn Sīnā's mathematical ideas among the Latin Schoolmen.

A title by title examination of Anawati's bibliography yields 8 works of Ibn Sīnā which seem to have mathematical content. We consider them

was primarily a biographer and historian. See, for example, Brockelmann's *GAL*, I (supplement), 557-558.

<sup>19</sup> See also Wiedemann, E. and Juynboll, T. W., "Avicennas Schrift über ein von ihm ersonnenes Beobachtungsinstrument." *Acta Orientalia* 5 (1927), 81-167, and Wiedemann's "Über ein von Ibn Sīnā (Avicenna) hergestelltes Beobachtungsinstrument," *Zeitschrift für Instrumentenkunde* 45 (1925), 269-175. These are translations and commentaries on the manuscript (Leiden, Or. 184/8): *Maqāla fi l-Ālāt arrāṣadiya*.

<sup>20</sup> Gohlman (38), 148 and 152. The references are to Y. Mahdavi : *Fihrist-i muṣannafāt-i Ibn-i Sīnā*, Publications of the University of Tehran, no. 206, Tehran, 1954, and G. C. Anawati : *Mu'allafāt Ibn Sīnā*, Cairo, 1950. See also Osman Ergin : *Ibni Sina Bibliografyasi*, İstanbul, Osman Yalçın Matbası, 1951, XXXI, 168p (İstanbul Üniversitesi Tıp Fakültesi yayınlarından, 20). There is a critical review by Rudolf Sellheim in *Orins* 11 (1958), 231-239.

almost in the order he adopts. The first is given the number 13.: Danish nāmé ‘Alā’ī<sup>21</sup>, is in Persian and contains seven parts: logic, metaphysics, physics, geometry, astronomy, arithmetic and music. Part is the work of Ibn Sīnā, while the rest, primarily the mathematical section, is a translation of writings of Ibn Sīnā into Persian by al-Jūzajānī after his master’s death. In fact, in the preface to the section on geometry, al-Jūzajānī claims that the original section on mathematics was lost, and that he replaced it with another.<sup>22</sup> Now al-Jūzajānī also composed prefaces for the mathematical sections of the Najāt, and these differ little from those of the ‘Alā’ī.<sup>23</sup> In Najāt he asserts that the mathematical sections are assembled from other writings of Ibn Sīnā, and that the arithmetic section is extracted from the Shifā’.<sup>24</sup> These mathematical parts of Najāt do not appear in the Cairo edition of Ibn Sīnā’s work, but are preserved in manuscript, e. g. Ahmet III, no. 34481, Istanbul. Mahdavi has shown that the mathematical sections of ‘Alā’ī are merely a Persian translation of the corresponding Arabic text of Najāt.<sup>25</sup> Nonetheless, al-Jūzajānī himself remarks that the Najāt consists of three books-probably the logic, physics and metaphysics-which Goichon<sup>26</sup> has shown are mostly extracts from the Shifā’. In fact, he states that the original ‘Alā’ī consisted of “one volume.”<sup>27</sup>

Al-Najāt, 23 in Anawati’s bibliography,<sup>28</sup> exists in at least 28 manuscripts, and al-Jūzajānī tells us that Ibn Sīnā composed it “en route”, “in the year ‘Alā al-Dawla attacked Sābūr Khwāst.”<sup>29</sup> Gohlman discusses the problem of exact date, noting that al-Dawla attacked Sābūr several times, and decides in favor of 1030 or 1032.<sup>30</sup> Al-Jūzajānī adds that the ‘Alā’ī

<sup>21</sup> Ergin 74. See Anawati (note 20) for actual locations of manuscripts. Carra de Vaux (Avicenna, 147. n. 3) identified Danesh name’-i ‘Alā’ī with Anawati 11 : Hikma ‘alā’ iyya

<sup>22</sup> Le Livre des science II, 91.

<sup>23</sup> Le Livre des science II, 6.

<sup>24</sup> *Ibid.*, 6.

<sup>25</sup> Mahdavi : Fihrist-i muṣannafāt-i Ibn-i Sīnā, 101 and 226.

<sup>26</sup> Anawati (8), 419, or A. M. Goichon : La Distinction de l’essence et de l’existence d’après Ibn Sīnā (Avicenne), Paris, 1937; 499-503.

<sup>27</sup> Gohlman (38), 47.

<sup>28</sup> See also no. 24 in Anawati’s list : al-Hidāya, msa. Aya Sofya 24775\*1, also in 3 parts logic, physics and metaphysics.

<sup>29</sup> Gohlman (38), 67.

<sup>30</sup> Gohlman (38), 132-133; n 89.

was written in Isfahan,<sup>31</sup> a fact confirmed by the anonymous bibliography appended to *Sīrat*.<sup>32</sup> This would place its composition somewhere between 1024 and 1037. But since our interest is in the mathematical sections this has little relevance. *Stili*, the preceding discussion does establish that the Arithmetic section of ‘*Alā’ī*’ derives from *Najāṭ*, which, in turn, has its source in the *Shifā’*, and that the geometric sections of ‘*Alā’ī*’ and *Najāṭ* are the same. In a later section, we shall compare their geometric content with that of the *Shifā’*.

Besides the *Shifā’* itself, *Anawati* lists 5 other works of possible mathematical content. His no. 75 : *al-nihāyawal-lānihāya* (the finite and the infinite) is really a discussion of the question : does time have a beginning, and so, will be dismissed from further consideration.<sup>33</sup> Number 160 : *al-Zāwiya* (the angle), or *taḥqīq mabādī’ al-handasa*<sup>34</sup> was probably written in *Jurgān* between 1012-1014,<sup>35</sup> and discusses the  $\chi\epsilon\rho\alpha\tau\omicron\epsilon\iota\delta\eta\varsigma$  or “hornlike” angle between a circle and its tangent. This was a problem mentioned by Aristotle<sup>36</sup> and settled completely in *Euclid*.<sup>37</sup> Number 161 : *Ḥall mushkilāt mu’ayyana* (Method of Solving Problems) [Persian] exists in 1 ms, *Aya Sofya* 4853\*25. We can offer no information on its possible content. The other two items, no. 169 and 170 are respectively, extracts from the *Shifā’*: *Mukhtaṣar Oklidos* and *Mukhtaṣar Kitāb al-arithmātiqī*.<sup>38</sup>

The *Shifā’*, the evident source of much of *Ibn Sīnā*’s “other” mathematical writing, is preserved in over 100 ms.<sup>39</sup> *Al-Jūzajānī* tells us, as noted earlier, that it was begun possibly with the physics section, shortly after book I of *al-Qānūn*.<sup>40</sup> The anonymous bibliographer claims that both the physics and

<sup>31</sup> Gohlman (38), 69 and 97.

<sup>32</sup> Gohlman (38), 69 and 97.

<sup>33</sup> *Anawati* (8) 424. *Ibn Sīnā* argues that time has no beginning.

<sup>34</sup> *Anawati* (8), 431.

<sup>35</sup> Gohlmann (38), 99. The author of the anonymous bibliography records this as “a letter to *Abū Sahl al-Masīhī*” written in *Jurjān*. But see also Gohlman (38), 139, n. 5, and 124-125, n. 43.

<sup>36</sup> See Heath (46) I, 177-178; Aristotle, *Analytica Priora* I 24 : 41b, 13-22 (Oxford ed. vol I) and *Elements* III, 16.

<sup>37</sup> See Heath (46) I, 177-178; Aristotle, *Analytica Priora* I 24 : 41b, 13-22 (Oxford ed. vol I) and *Elements* III, 16.

<sup>38</sup> *Anawati* (8), 432.

<sup>39</sup> *Anawati* (8), 417.

<sup>40</sup> Gohlman (38), 55.

metaphysics sections were done in twenty days in Hamadhān.<sup>41</sup> This would be after 1015. The logic section was partly done there,<sup>42</sup> and later completed in Isfahan after 1024.<sup>43</sup> Sections on arithmetic, geometry and music seem to have been worked out before then,<sup>44</sup> while the section on astronomy may have taken shape when Ibn Sīnā was in Jurgān (1012-1014).<sup>45</sup> Al-Jūzajānī also seems to indicate that the mathematical parts of al-Shifā' were expanded and polished during Ibn Sīnā's stay in Isfahan.<sup>46</sup> The text that comes down to us consists of four parts : logic, physics, mathematics and metaphysics, in around 430 folios, with the subjects being allotted roughly 43, 32, 16 and 9 percent of the text, respectively. The 70 folios devoted to mathematics are split between geometry, arithmetic, music and astronomy, with roughly 30 percent to geometry, 50 to astronomy and the remaining 20 distributed with slight favor toward music<sup>47</sup>. The astronomy section also contains material on ratios and on trigonometry.

In addition to these works of evident mathematical content, noted from surveying Anawati's bibliography, two others give further information : the *als-As 'ilah wa'l-Ajwibah* (Questions and Answers)<sup>48</sup> exchanged between Ibn Sīnā and al-Bīrūnī, and al-Bīrūnī's *Kitāb Taḥdīd al-Amākin li-taṣḥīḥ masāfat al-masākin* (Determination of the Coordinates of Cities).<sup>49</sup> The former offers some insight into Ibn Sīnā's notion of infinity,<sup>50</sup> while the

<sup>41</sup> Gohlman (38), 92-93.

<sup>42</sup> Gohlman (38), 61.

<sup>43</sup> Gohlman (38), 65.

<sup>44</sup> Gohlman (38), 65.

<sup>45</sup> Gohlman (38), 45.

<sup>46</sup> Gohlman (38), 65-66.

<sup>47</sup> These figures are derived from Anawati's description of the al-Azhar ms. of the *Shifā'*; Anawati (8), 418.

<sup>48</sup> S. H. Nasr and M. Mohaghegh: *Al-Bīrūnī and Ibn Sīnā: als-As 'ilah wa'l-Ajwibah*, Tehran 1973. There is also a Russian edition: I.M. Mumizov, ed., *al-Bīrūnī and Ibn Sīnā Correspondence* (Russ.), Izdat. "Fan," Uzbek SSR, Tashkent, 1973. This is a translation of ms. 2385, Arch. Inst. Far Eastern Studies, Uzbek Acad. Sci. Parts of these "questions and answers" are translated into English by Nasr in (82), 133-138. For additional manuscripts, see Anawati (8), 415. no. 1-2 : 422, no. 54.

<sup>49</sup> There is a Russian edition by P. G. Bulgakov (with commentary) : *Abu Reihan, Izbrannye Proizvedeniya III (Geodeziya)*, Akad. Nauk. Uzbek. SSR. Tashkent, 1966. The English translation by J. all (5), is supplemented by Kennedy's commentary (57).

<sup>50</sup> Certain passages of the metaphysics of al-Najāt are also relevant. Sec § VI of text.

latter contains a criticism by al-Bīrūnī of an astronomical calculation in a lost work of Ibn Sīnā.<sup>51</sup>

Perhaps we should mention two further manuscripts: the first is a poem in praise of Euclid attributed to Ibn Sīnā by Ūnver;<sup>52</sup> (a translation appears in the notes.) The second, al-nīrūziyya fī ḥurūf abjad (risālat al-ḥuruf) has some connection with a magic square of order three, but Ibn Sīnā's interest is cosmological and not mathematical, so we need not discuss it.<sup>53</sup>

Of course, we cannot claim to have exhausted all possible references of Ibn Sīnā to mathematics. For example, in his Aristotelian commentary al-Burhān,<sup>54</sup> he states that the solvability of  $x^3 + y^3 = z^3$  in integers is still unresolved in his time!

Steinschneider's classic study of European translations of Arabic manuscripts indicates that although Ibn Sīnā's Canon of Medicine (al-Qānūn fīl ṭibb) was well known to the Latin West in the late Middle Ages, the mathematical sections of the Shifā' were not<sup>55</sup>; in fact it is only the (pure) logic section that was fully translated.<sup>56</sup> This and parts of the physics had considerable impact on European science in the time of the

<sup>51</sup> Kennedy (57), 124-125; al-Bīrūnī (5), § 65.

<sup>52</sup> Ūnver (116), 199. His free rendering reads: "Euclid in science is so great that his name is synonymous with events happening in heaven and on the horizons. He is like a stairway whose steps are his figures, on which ascend passengers to endless heights. A person who makes his acquaintance is sure to rise to a high level. That level is ever so pleasant and the person who reaches it is ever so happy." However, see Murdoch (78), 453, note 4.

<sup>53</sup> Anawati (9), p. 422, no. 49. This is part of a nine essay collection: *Tis rasā'il*, Cairo, Hindīyah Press, 1908. For partial translation into English, with commentary, see Nasr (81), 209-212.

<sup>54</sup> See Anboubā (10), 92 and al-Burhān, ed. Badawī (Cairo, 1954), 135.

<sup>55</sup> Steinschneider (104), II, 92, etc. Even the printed editions of the Shifā' done in the early 16th century are merely selections. For example, *Avicene ... philosophi : ac medicorum facile primi opera in luce redacta ...*, Venice, 1508 contains the Logic (Book 1, fen 1, makālah 1, 2 translated by Gundisalvus); *Physica* (Book 2, fen 1, makālah 1, 2 and faṣl 1, 2 of Makālah 3, translated by Gundisalvus); *De Anima* (kitāb al-naḥs) (Book 2, fen 6 translated by Hispalensis and Gundisalvus); *De animalibus* (al-hayawān) (Book 2, fen 8 translated by Michael Scott); *Philosophia prima* (Metaphysics, i. e. Book 4, as well as sections titled "De intelligentiis. Alfarabius de intelligentiis," and "De celo et mundo," which the descriptive catalogue (B. M. Arabic Books I, 664) lists as a collection based on Aristotle, thereby distinguishing it from al-sam ā' wal 'ālam (Shifā', *Physica*, fen 2).

Apparently al-Najāt also enjoyed some currency; for example, the 1593 Rome edition (Arabic) of al-Qānūn fī al-ṭibb contains it as an appendix. (See B. M. Arabic Books I, 670, 683).

<sup>56</sup> Anawati (8), 417.

Latin Schoolmen (13th-14th centuries). Haskins,<sup>57</sup> Rashdall<sup>58</sup> and Crombie<sup>59</sup> discuss this impact: the first, in terms of early translators; the second, in the setting of the Medieval university; and Crombie, in the works of various scientists, including Robert Grosseteste, Roger Bacon, Albertus Magnus and Leonardo da Vinci. In addition, Alverny's extensive series "Avicenna Latinus,"<sup>60</sup> adds much to Steinschneider's work, and is invaluable to further study of Ibn Sīnā's influence in Medieval Europe. Finally, the studies of Goichon and Madkour cast additional light on the impact of his works in pure logic.<sup>61</sup>

Although Ibn Sīnā's writings in mathematics had no impact on the Latin West, and probably little influence among Near Eastern mathematicians, his mathematical writings, particularly from the *Shifā'*, do receive citation in the Arabic literature of the Middle Ages. We have already mentioned the remarks of al-Jūzajānī and Ibn Funduq. Later, we shall give citations (rather uncomplimentary) from al-Bīrūnī and al-Maghrabī. More typical, however, are those references made by Ibn Khaldūn in the *Muqaddimah*. In sections 19-21, he notes Ibn Sīnā's contributions in arithmetic, geometry and astronomy, neglecting only trigonometry (Menelaus' theorem).<sup>62</sup>

Having detailed Ibn Sīnā's mathematical education, and listed his mathematical texts, let us consider them in the next four sections: arithmetic, geometry, trigonometry, and foundations. This last division, of course, is not a special heading of al-*Shifā'*, 'Alā'ī, or any of the mathematical essays, but we include it as a catch-all for discussing Ibn Sīnā's position on infinity, infinitesimals, limiting processes, and other issues in the foundations of mathematics.

<sup>57</sup> Haskins (42), 13, 245, 360)

<sup>58</sup> Rashdall (86);i, 236n., 353, 365, 436; ii, 33n, 127; iii, 245n.

<sup>59</sup> Crombie (24), 26, 72, 90, 92-93, 100, 370; and Crombie (26), 79-81, 324.

<sup>60</sup> Alverny, M.-Th. d', "Avicenna latinus" in Arch. hist. doct. et litt. du moyen âge, I, t. 28 (1961), 281-316; II, t. 29 (1962), 217-333; III, t. 30 (1963), 221-272; IV, t. 31 (1964), 271-286; V, t. 32 (1965), 257-302; VI, t. 33 (1966), 305-327; VII, t. 34 (1967), 315-343; VIII, t. 35 (1968), 301-335. See also M. Alverny's "Les traditions manuscrites de l'Avicenne latin," in *Mélanges Taha Hussein*, Cairo, Dār al-Ma' āref, 1962, 67-75.

<sup>61</sup> For example : A. M. Goichon: *Lexique de la langue Philosophique d'Ibn Sīnā*, Paris, 1938, etc., and I. Madkour : *L'Organon d'Aristotle dans le monde arabe*, Paris, 1934.

<sup>62</sup> Rosenthal (59), III, 118-134, or Woepcke (123).

### III. Arithmetic

As we have noted already, Ibn Sīnā's writing in arithmetic derive primarily from the *Shifā'*, with a line of dependence reaching from the 'Alā'ī via the *Najāṭ* to the *Shifā'* itself. Our discussion of the arithmetic will concentrate first on the 'Alā'ī, and then shift to the *Shifā'*, which has recently appeared in an Arabic edition of Lutfy.<sup>63</sup> Because of the intimate relation between 'Alā'ī and *Najāṭ*, we give no special consideration to the latter. As will become apparent, even a survey based on the 'Alā'ī itself gives ample evidence of the content of Ibn Sīnā's arithmetic, and omits little of interest in the *Shifā'*, apart from certain sections on "speculative arithmetic."<sup>64</sup>

In his introduction to the arithmetic of the 'Alā'ī, al-Jūzajānī remarks that he includes only that part of Ibn Sīnā's writing essential to the study of music.<sup>65</sup> The section is in seven parts, and we now offer a summary of each. Our purpose is twofold : first, to give the reader a clear notion of the contents, and second, to determine the sources. We use formulas for brevity only; Ibn Sīnā's arithmetic is purely rhetorical. The impatient reader may skip ahead to our conclusions, returning to the summary as needed.

#### § 1. Characteristics and Kinds of Numbers

The author begins by defining number as a collection of units, and separates them into two kinds : the even, which are divisible into two equal parts, and odd, which are not. He then notes that every number except 1 is the mean between two numbers equally distant from it, e. g. 5 is the mean of 6, 4; 7, 3; 8, 2; 9, 1. He defines the median of the sequence 1, ..., n for n odd to be the average of 1 and n, and notes that it is the middle term, but for n even, recognizes two medians, e. g. 5 and 6 for 1, ..., 10, recognizing that  $5 + 6 = 10 + 1$ , and the numbers equally spaced about the median as 7 and 4 have the same property.

The formula  $1 + \dots + n = \frac{n(n+1)}{2}$  appears as  $1 + \dots + n$  equals half the number of elements times the number of elements plus one. Then it is noted

<sup>63</sup> The *Arithmetic and Geometry* appeared in 1970 and 1975 respectively, see (101) and (102).

<sup>64</sup> See also : Eneström (32), Montferrier (76), Tannery (110) and Woepcke (122).

<sup>65</sup> Achena, Massé edition (100), 190.

that  $1 + \dots + n + (n - 1) + (n - 2) + \dots + 1 = n^2$ , with  $n = 10$ , but with the implication that the result is perfectly general. Curiously enough, there is no correlation in the text between this and the statement a few paragraphs later remarking that the sum of the odd numbers  $1, 3, 5, \dots, 2k + 1$  is a square.

The author then suggests another addition formula for  $1 + \dots + n >$  by observing the pattern :  $1 + 2 = 3 = 2 + \frac{1}{2} (2)$ ;  $1 + 2 + 3 = 6 = 2 (3)$ ;  $1 + 2 + 3 + 4 = 10 = 2(4) + \frac{1}{2} (4)$ ;  $1 + 2 + 3 + 4 + 5 = 15 = 3(5)$ ;  $1 + \dots + 6 = 21 = 3(6) + \frac{1}{2} (6)$ , etc. The section closes with the observation that odd numbers are not products of even numbers, and not “even” sums of even numbers, and the statement concerning sums of odd numbers given above. Apart from squares there is no reference to the other Pythagorean numbers.

## § 2. Even Numbers

In the second part, devoted to even numbers, the author notes that  $2 + 4 + \dots + 2n = n^2 + n$ , using  $n = 3$  as the generalizable instance. He then lists three kinds of even numbers : 1) evenly even, 2) evenly odd, 3) oddly even. The first kind have only the divisor 2 (and its powers), such as 64, and hence no odd factors. If an evenly even number  $n = q(r)$ , then both  $q$  and  $r$  are also evenly even. Such numbers are generated as a sequence  $1, 2, 4, 8, \dots, 2^n, \dots$ . There is also a median property which is best illustrated by the following self-explanatory diagram. Terms equidistributed (geometrically) about the median (or medians) have the same product and are called by the author, complimentary. (See figure 1.) The translation, as given by Achena and Massé, then asserts that  $1 + 2 + 4 + \dots + 2^n < 2^{n+1}$ , an unlikely reading<sup>66</sup> given that the correct formula  $1 + 2 + \dots + 2^n = 2^{n+1} - 1$  appears in Euclid IX,

<sup>66</sup> *Ibid.*, 194.

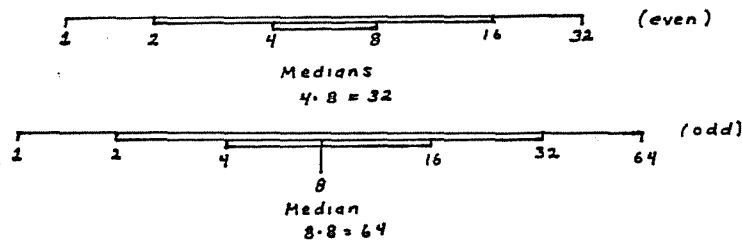


Figure 1

prop 35,<sup>67</sup> and that the author shortly remarks that every evenly even number is deficient with deficiency 1!<sup>68</sup> The author records other properties of these numbers : in the sequence  $1 \dots, 2^n, 2^k$  divides  $2^{k+1}$ , that is, “successive members of the sequence measure each other” in the sense of Euclid; the squares of such numbers belong to the same class;  $2^n - 2$  is evenly odd.<sup>69</sup> The latter term is yet to be defined in the text. Then comes the remark on the deficiency of  $2^n$ ; deficient perfect and abundant numbers are defined later. This is followed with a paragraph which notes that successive evenly even numbers stand in geometric ratio, a more precise form of the “measure” statement.

Now there follows a rule for using these numbers to obtain perfect numbers: add  $1 + \dots + 2^n$  to a place where the sum  $[2^{n+1} - 1]$  is prime, multiply this sum by  $2^n$ . The product is then perfect. The rule is illustrated by  $6 : 1 + 2 = 3$ , a prime and  $2(3) = 6$  is perfect, and by  $28 : 1 + 2 + 4 = 7$ , prime and  $4(7) = 28$ .

Evenly odd numbers are then defined as those having only a single factor of 2, e. g. 6, 10, and are obtained by multiplying successive odd numbers by 2. Successive members of evenly odd sequences thus differ by 4, and no such number can be a square or cube, because squares or cubes are either odd numbers divisible by odd numbers, or even numbers divisible by even numbers, the quotients being the same [in parity] as the divisor. Oddly even numbers have the divisor 2 more than one, “but don’t end in one,”<sup>70</sup> i.e. aren’t successively divisible by 2 as are evenly even. The author conceives

<sup>67</sup> Shifā’ (Cairo edition of Geometry), 293-294, no. 35.

<sup>68</sup> *Ibid.*, 194.

<sup>69</sup> *Loc. cit.*

<sup>70</sup> *Ibid.*, 195.

of them as between evenly even and evenly odd. They arise by multiplying odd numbers by evenly even numbers. He then shows by examples that oddly even numbers may be deficient, perfect, or abundant, e.g.

$68 = 4 (17) > 1 + 2 + 4 + 17 + 34$  (deficient),  $28 = 4 (7) 1 + 2 + 4 + 7 + 14$  (perfect),  $12 = 4 (3) < 1 + 2 + 3 + 4 + 6 = 16$  (abundant).

### § 3. Odd Numbers

In the third section the author notes that there are three kinds of odd numbers : the primes (first non-composite); the second composite, which are products of two odd numbers (excluding 1 as a factor); the third composite. These are actually pairs of numbers of the second composite class that are relatively prime, e. g. 9 and 25. The primes have no factors but themselves and unity. Examples are 3, 5, 7, 11. The text then reads “All other numbers are composed of these prime numbers, whereas, they themselves are not reducible.”<sup>71</sup> Evidently the author either intends “all other numbers” to be the odd ones since 2 is not a prime in his sense, or, more likely, that the evenly even numbers provide the factorizations by powers of 2. (See previous section). There is no statement of the number of primes, and no explicit mention of “unique factorization.” This part ends with a method for finding primes and second composite numbers : consider a square table of successive odd numbers, and “do as other scholars have shown.”<sup>72</sup> Presumably this is a reference to a sieve-method.

### § 4. [Abundant, Deficient and Perfect Numbers]<sup>73</sup>

In section four the author states that abundant numbers are those whose proper divisors, in sum exceed the number. 12 is an example. Deficient numbers have the sum of their proper divisors less than the number, e. g. 8; and perfect numbers, like 6, are equal to the sum of their proper divisors. He then remarks that it is necessary to know that a number can be perfect only if it is the product of an even and odd number.<sup>74</sup> Now among the first 9, there is but one, 6; among the first tens, only 28; between 100 and 1000 only

<sup>71</sup> *Ibid.*, 197.

<sup>72</sup> *Ibid.*, 198.

<sup>73</sup> Title added by translators.

<sup>74</sup> *Ibid.*, 198.

496, and between 1000 and 10000 only 8128.<sup>75</sup> Next he gives a prescription for finding perfect numbers : multiply an evenly even number by a prime number, subject to the condition that the evenly even number exceeds by one half unit, half the prime. [That is  $2^n - \frac{1}{2} = \frac{P}{2}$  or  $P = 2^{n+1} - 1$ , the condition of sections 2]. If the evenly even number exceeds by more than one half unit, half the prime, the resulting number is abundant, and if it is less by one unit than half the prime, the resulting product is deficient.<sup>76</sup> Examples are 2 (3) and 4 (7) (perfect), 4 (5) (abundant); 4 (9) is cited as deficient, [however, it is abundant. Moreover 9 is not prime and the rule does not apply as  $4 + \frac{1}{2} = \frac{9}{2}$  ]. This is a curious oversight. However, the author does supply an example with 4 (11).<sup>77</sup>

### § 5. [Ratio]

In section five, Ibn Sīnā notes that there are different meanings attached to the term ratio; for example, there are ratios which compare size of numbers [the sense adopted here], and ratios which compare qualities like kind and class. Having made this point, he states a trichotomy law, and an inversive principle for ratios : of two different numbers, one is the larger and the other the smaller; when one knows the ratio of the larger to the smaller, one knows, inversely that of the smaller to the larger.

Ratios are classified as simple or composite; simple, if the larger is twice, or some greater multiple of the smaller, or is a unit plus a part or many parts of the smaller;<sup>78</sup> composite, if the larger exceeds the smaller by twice and a part, etc. The first simple ratio is the double, in which the excess of the larger over the smaller is equal to the smaller, as 1 and 2. The next type, the triple, has the larger 3 times the smaller, as 1 and 3. Letting the smaller term 1 “run through the natural numbers,” while the larger goes through their triples generates such ratios, e. g.,  $\frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \frac{12}{4} \dots$  The next type of ratio has the larger exceed the smaller by the mean of the smaller, e.

<sup>75</sup> *Ibid.*, 199.

<sup>76</sup> *Ibid.*, 199.

<sup>77</sup> *Ibid.*, 199.

<sup>78</sup> *Ibid.*, 199.

g. 2, 3, since  $\frac{3}{2} = 1 + \frac{1}{2}$ , To generate other ratios of this type 3 goes through the triples of the natural numbers and 2 through the doubles:  $\frac{3}{2}, \frac{6}{4}, \frac{9}{6}, \dots$  The remainder of this part takes up, in order, analogous discussions for the ratio  $1 + \frac{1}{3} : 1$ , i. e. 3 and 4;  $1 + \frac{2}{4}$ , i. e. 4 and 6, etc. The author wishes to indicate how to get the ratios in lowest terms, noting that ratios of the form one plus two or more parts may either be reduced or not.<sup>79</sup> As examples:  $1 + \frac{2}{6}$ , i. e. 6 and 8 are not, but  $1 + \frac{2}{5}, 1 + \frac{3}{4}$ , i. e. 5 and 7, 4 and 7 are. The general rule is : take a number equal to the *part* as the smaller number, and add it to the numerator e. g.  $1 + \frac{1}{3}; 3 + 1$ , so 3 and 4, if there are 2 parts, add it to 2, e. g.  $1 + \frac{2}{3}; 2 + 3$ , so series of 3, 5.

The author then takes up the rules for composites by example; thus  $2 + \frac{1}{2}$ , gives 2 and 5,  $2 + \frac{1}{3}$ , 3 and 7;  $2 + \frac{1}{4}; 2 + \frac{2}{3}; 2 + \frac{2}{4}; 2 + \frac{2}{5}$ .<sup>80</sup> Thus to get the series for  $2 + \frac{1}{4}$ , he multiplies 4 and 2 and adds one, giving 4 and 9. This continues to  $3 + \frac{1}{4}$ , to get 4 and 13, and concludes with the observation that in the triples and a part, the small terms proceed by one, while the larger terms grow by three, that is  $3 + \frac{1}{2}, 3 + \frac{1}{3}, 3 + \frac{1}{4}$ , etc. yield 2, 7; 3, 10; 4, 13. The same analysis was applied earlier, in the text, to the unit and a part, a double and 2 parts.

### § 6. [Composition of Ratios]

In this section the composition of ratios is essentially their products, but such a statement does violence to the actual method of the author, since ratios are never thought of as fractions. This is best illustrated by Ibn Sīnā's own examples. As usual he observes that each particular example is representative of a whole class, following this with an example to show that the ratio of  $1 + \frac{1}{3}$  composed with  $1 + \frac{1}{2}$  is 2 : Let AB = 4, AC = 2, AD = 3.

<sup>79</sup> *Ibid.*, 202.

<sup>80</sup> *Ibid.*, 203-206.

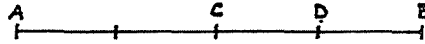


Figure 2

Then he notes that  $AB$  to  $AD = 1 + \frac{1}{3}$  to 1,  $AD$  to  $AC = 1 + \frac{1}{2}$  to 1 and so  $AB$  to  $AC$ , the “composition” is the “double.” Moreover, he argues that this is perfectly general, in the sense that actual integral values of  $AB$ ,  $AC$  and  $AD$  are irrelevant; only the ratios have bearing. In fact he asserts that any double can be decomposed into a composition of a  $1 + \frac{1}{3}$  to 1 and a  $1 + \frac{1}{2}$  to 1. The simple argument establishing the latter, essentially reconstructs a figure similar to  $ABCD$  (above) using only ratios, identifies corresponding points, asserts that corresponding segments all lie in the same ratio to each other, and hence by equality and composition decomposes the “double” of the similar figure in the same manner as  $ABCD$ .<sup>81</sup>

There follow a number of such numerical examples. A few will suffice to give the flavor of the work.

a) Let  $CD$  be a third of  $AC$ , and  $AC$  twice  $AB$ , then  $AD$  is the double of  $AB$  plus a third of  $AB$ .  $\left[ \frac{CD}{AC} = \frac{1}{3} \text{ and } \frac{AC}{AB} = 2 \frac{AD}{AB} = 2 + \frac{2}{3} \right]$ . For the proof, divide  $AC$  into

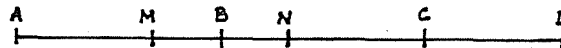


Figure 3

three parts, by  $M$  and  $N$ . Then  $AM$  must be equal to  $CD$ , for the latter is also a third of  $AC$ , and  $AC$  is the double of  $AB$ . Now the mean of  $AM$  is a third of  $AB$ , consequently  $AM$  must be two thirds of  $AB$ . Thus  $AD = AC + CD = 2 AB + \frac{2}{3} AB$ .<sup>82</sup>

b) Or again, if  $AC$  to  $AB$  is one plus one third, and  $BC$  to  $BD$  is one plus one half, then  $AB$  to  $BD$  is one to one half. For the proof, divide  $AB$  into three parts by

<sup>81</sup> *Ibid.*, 206-209.

<sup>82</sup> *Ibid.*, 206-209.

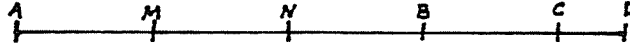


Figure 4

the points M and N. Then AM, MN, NB and BC are equal, and the mean of each of these is an eighth part of AC, and must equal CD. Thus BD is three times CD, and AB is six times CD, which is the ratio of one to one half. Moreover, the ratio of BC to CD is the same as that of AB to BD, or equivalently BD to AB equals DC to BC, which is the ratio of the same to half the same.<sup>83</sup>

C) Similar constructions are produced to show that if AC to AB is 5 to 4, and AC to CD is 6 to 5, AD to AB is 3 to 2; AC to AB 6 to 5 and AC to AD 7 to 6 imply AD to AB is 7 to 5. In addition, there are arguments to show that  $\frac{7}{6} \cdot \frac{8}{7} = \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{10}{9} = \frac{5}{4} \cdot \frac{10}{9} \cdot \frac{11}{10} = \frac{11}{9} \cdot \frac{11}{10} \cdot \frac{12}{11} = \frac{6}{5} \cdot \frac{15}{14} \cdot \frac{16}{15} = \frac{8}{7}$ .<sup>84</sup> The last demonstration of this section should show that if AC to AB is 6 to 5 and AD to AC 5 to 4, then AD to AB is 3 to 2, but text is garbled and the result is twice stated as 4 to three, i.e. one and one third instead of one and one half. The author closes with the assurance that this suffices for the proofs of all assertions in the section on music.

## § 7. [Proportions]

Al-Jūzajānī begins this section by remarking that he has omitted some notes of Ibn Sīnā, namely the definition of integers, and their quantitative aspects. He then mentions offhandedly, that there are 20 kinds of proportions, 6 of which are useful to music theory.<sup>85</sup> In particular, there are proportions depending on the notion of surplus: one speaks of the surplus of a over b being equal to the surplus of c over d, i. e. the difference of a and b equal to the difference of c and d. Another sort of proportion he calls geometric; this involves a comparison of ratios, or in modern notation  $a/b = c/d$ . He notes that these are, in general, different; for example, 6 is to 4 as 9 is to 6, but 6

<sup>83</sup> *Ibid.*, 206-209.

<sup>84</sup> *Ibid.*, 210.

<sup>85</sup> *Ibid.*, 211, Actually there are 10.

– 4#9 – 3; while an arithmetic proportion 4, 6, 8 where  $6 - 4 = 8 - 6$  is not geometric, since  $\frac{6}{4} \neq \frac{8}{6}$ .

Next the author asks us to look at numbers with composition such that the ratios between them are successive, with the greatest ratios being those between the largest numbers, and the smallest ratios those between the smallest numbers. For example 6, 4, 3 gives  $\frac{6}{4} = 1 + \frac{1}{2}$  and  $\frac{4}{3} = 1 + \frac{1}{3}$ . He calls these ratios of composition because of their use in musical composition, and, possibly, also because the ratio of the two extremes equals the ratio of the surpluses.  $\left[ \frac{6}{3} = \frac{6-4}{4-3} \right]$ ; this actually defines the harmonic mean].

The author next takes up the study of the geometric mean, whose square is the product of the extremes. Similarly, if there are two means, their product equals the product of the extremes.  $\left[ \frac{a}{m} = \frac{m}{b}, \frac{a}{m_1} = \frac{m_2}{b} \right]$  give  $m^2 = ab, m_1 m_2 = ab$ . On the basis of this he notes that a three term geometric series is constituted by interposing consecutive rectangular numbers between consecutive squares. [The allusion is apparently of the sort:  $\frac{a^2}{x} = \frac{y}{b^2}$  implies  $xy = (ab)^2$  a rectangular number, “interposed” between squares,  $a^2$  and  $b^2$ ]. Also he states that between two squares there can be only one geometric mean, and between cubes there can be two.<sup>86</sup>

In the case of arithmetic proportions, the text continues, the numbers grow uniformly, be it by units, or by tens, etc. They are continuous with respect to a single mean, but not continuous with respect to two. This mean is determined by the extremes. When two elements stand in arithmetic proportion, all is known. The mean is just the average of the extremes, and has the property that the product of the extremes is less than its squares. [Given  $a, a + d, a + 2d, (a + d)^2 > a(a + 2d)$ ]. “And the measure of this difference is just the square of the excess of one extreme over the other.”<sup>87</sup> [Or, so the text reads in Achena-Massé’s version. Actually  $(a + d)^2 = a(a + 2d) + d^2$ , i. e. The measure of the difference is just the square of the excess

<sup>86</sup> *Ibid.*, 214.

<sup>87</sup> *Ibid.*, 215, “Son caractère propre consiste en ce que le produit de la multiplication des deux extrêmes est moindre que le carré du moyen, et cela à la mesure du carré de l’excès de l’un des deux extrêmes sur l’autre.”

of either extreme over the mean. That the translation is merely garbled here is indicated by Ibn Sīnā's clear illustration: for 2 and 6 the mean is 4, and  $4^2 - 2(6) = 4 = 2^2$ .

As to composite proportions [ratios of composition], the author observes that we know that they differ from arithmetic proportions, and how. To obtain the mean term between the extremes  $a$  and  $b$ , ( $a$  less than  $b$ ), take for it  $a$  times the excess of  $b$  over  $a$ , divided by the sum of  $a$  and  $b$ , and add this quantity to  $a$ . This is the mean; [i. e.  $a(b-a)/(a+b) + a = \frac{2ab}{a+b}$ ]. For example the mean between 6 and 3 is  $3 + 3(6-3)/(6+3) = 4$ . If the mean and the larger term are known, then the smaller term is  $m - m(b-m)/(2b-m)$ .<sup>88</sup> [The translation of Achena and Masse, however, gives a nonsense expression "obtenir la différence entre le puis divisor le résultat par le moyen terme *une seconde fois* et en deduire enfin le résultat du moyen terme; c'est alors le petit terme."<sup>89</sup>] When only the smallest term  $a$  and the mean are known, then the largest term is  $\frac{m}{\frac{a}{m-a} - 1} + m$ . The

author then lists further properties of such composition proportions: if  $a$ ,  $m$ ,  $b$  are respectively the smallest, mean and largest term, then  $(a+b)m = 2ab$ ;  $mb = 2ma = 2ab$ . [The latter two equalities are, of course, nonsense, and may well be interpolations by another hand. The first  $mb = 2ma$  may have arisen when someone noted that Ibn Sīnā's example for this part 3, 4, 6 satisfies it, but this does not explain  $2ma = 2ab$ .]<sup>90</sup> Al-Jūzajānī closes this section by announcing the omission of seven other kinds of proportions, since Ibn Sīnā himself admitted that they have no application to music theory.

Having briefly sketched the contents of these seven sections of the arithmetic of the 'Alā'ī, let us note, at the outset, that the purported purpose of Ibn Sīnā's tract, to expose the reader to the mathematical elements necessary for studying music is amply met. The musical part of the 'Alā'ī requires nothing more in the way of mathematical introduction. This leaves

<sup>88</sup> *Ibid.*, 215-216.

<sup>89</sup> *Ibid.*, 215-216.

<sup>90</sup> *Ibid.*, 216.

us with the problem of establishing the possible sources of the arithmetic itself. Such sources will, of course, not be Greek originals, since he himself tells us that he knew no Greek.<sup>91</sup>

Ibn Sīnā's mathematical education gives us strong hints: the *Rasā'il* of the Ikhwān al-Ṣafā' contains a section on arithmetic, and so does the *Elements* (Books VII-IX). The former has a number of areas in common with Ibn Sīnā's work: division of even numbers into evenly-even, evenly-odd, oddly even, and division of odd numbers into prime and composite-odd, dropping the third class.<sup>92</sup> However; the remainder of the *Rasā'il*'s arithmetic contains no theory of ratios,<sup>93</sup> except a cursory mention in the geometry,<sup>94</sup> and some use in the musical theory.<sup>95</sup> A later section on the relationship between arithmetic, geometry and music<sup>96</sup> offers a few more details. But the *Rasā'il* provides a further possibility by referring to Nicomachus.<sup>97</sup> Much as the *Rasā'il* is unsuitable as the primary source of Ibn Sīnā's study, so is the *Elements*, as a perusal of Book VII-IX readily shows. So we turn to Nicomachus of Gerasa, who lived around 100 A. D., probably in the Gerasa of Palestine<sup>98</sup> His *Arithmetic* remained a popular mathematical handbook well into the 16th century in Europe, particularly among philosophers. An Arabic translation *Kitab al-Mudḥal ilā 'ilm al-'adad*, attributed to Thābit Ibn Qurra (836-901) survives.<sup>99</sup> Moreover the topics treated, as well as the order match Ibn Sīnā's arithmetic very closely. Nicomachus' *Arithmetic*, after a six chapter prologue to book one, defines number, and even and odd. (Chapter 7). Chapter 8 gives the median property, also treating one as exceptional; then lists the three classes of even numbers, and gives 64 as an example of an evenly even number. At this point Ibn Sīnā's text interpolates the formula for  $1 + 2 + \dots + n$  and  $1 + 2 + \dots + n +$

<sup>91</sup> Afnan (2), 190. Ibn Sīnā makes this admission in his commentary on Aristotle's *Poetics*, among other places.

<sup>92</sup> Dieterici (29), 11; see also Goldstein (40).

<sup>93</sup> Dieterici (29), *passim*.

<sup>94</sup> *Ibid.*, especially 33-34.

<sup>95</sup> *Ibid.*, 117 ff.

<sup>96</sup> *Ibid.*, 154 ff.

<sup>97</sup> *Ibid.*, 2.

<sup>98</sup> Tarān (111), 112.

<sup>99</sup> Our references will be to the edition of Kutsch (64).

$(n - 1) + \dots + 1$ , as well as the property, of which  $1 + \dots + 6 = 21 = 3.6 + \frac{1}{2}$  (6) is an example. None of these except the first (Book II, Chapter 9) occur in Nicomachus; nor does the formula for the sum of the first  $n$  even integers. Then Nicomachus gives the median property for evenly-even sequences, and the correct formula for  $1+2+\dots+2^n$ , remarking on its usefulness for constructing perfect numbers. Chapter 9 follows with the properties of evenly-odd numbers, while chapter 10 is devoted to oddly even numbers. The next chapter details the three classes of odd numbers; he also excludes even numbers from the class of composites.<sup>100</sup> The 13th chapter gives the sieve of Eratosthenes to which Ibn Sīnā only alludes, but there is also no reference to the number of primes. As one would expect, this is followed by a chapter on abundant, deficient and perfect numbers. Even the set of perfect numbers 6, 28, 496 and 8,128 appears, as well as the formula for perfect numbers  $2^n(2^{n+1} - 1)$  where the latter factor is a prime. However, Ibn Sīnā's criteria for abundant and deficient numbers are not in Nicomachus,<sup>101</sup> but he does assert that all perfect numbers are even, and in fact, ending in 6 or 8.<sup>102</sup> Nicomachus' examples of deficient numbers are 8 and 14, not Ibn Sīnā's 44. Chapter 18 of book one then introduces ratios, with a similar set of preliminary remarks, including trichotomy. Furthermore, ratios greater than one are divided into 5 groups : multiple, superparticular, superpartient, multiple superparticular and multiple superpartient, where the last four refer to  $1 + \frac{1}{n}$ ,  $1 + \frac{n}{m+n}$ ,  $m + \frac{1}{n}$ ,  $p + \frac{m}{m+n}$ . Ibn Sīnā calls the group consisting of multiples, superparticulars and superpartients simple, and the last two together complex. Nicomachus' discussion, paralleling Ibn Sīnā's construction of series for the various ratios occupies chapters 28-33 of book one.

Chapter 5 of book two of the Arithmetic handles the next section in Ibn Sīnā-composition of ratios. The first example is, in fact, concerned with the observation that the composition of  $\frac{3}{2}$  and  $\frac{4}{3}$  is 2. But the approach is different. While Ibn Sīnā proceeds geometrically, in a manner like that used

<sup>100</sup> D'Ooge (83), 202-203.

<sup>101</sup> D'Ooge (83), passim.

<sup>102</sup> D'Ooge (83), 210. As the editor notes, this remark also appears in Boethius, *De Institutione Arithmetica* I. 20 (Edition Friedlein, Leipzig, 1876).

in manuals like Al-Khwārizmī's Algebra to solve inheritance problems,<sup>103</sup> Nicomachus uses the mnemonic devices inherent in his classification scheme;

e. g. 2 sesquialter 3 sesquitercian 4,  
└────────── double ─────────┘

6 double 12 sesquilater 18.<sup>104</sup> Indeed, Nicomachus does not give any  
└────────── triple ─────────┘

evidence of concern for geometric justification; as he puts it “double with sesquialter is triple, triple with sesquitercian is quadruple, quadruple with sesquiquartan is quintile, and as far as you wish to proceed, no contrary result will appear.”<sup>105</sup> His chapters 6-20 of book two are devoted to Pythagorean topics: the figurate numbers, omitted by Ibn Sīnā. Finally, in chapter 21, he introduces Ibn Sīnā's last arithmetic topic, proportions, noting that six were known to the ancients, and ten in his own time.<sup>106</sup> In chapters 22-25 Nicomachus considers, in order, the arithmetic, geometric and harmonic (composition) proportions, noticing (chapter 24) that geometric and harmonic proportions are different, but with numerical example : 2, 4, 8.<sup>107</sup> Nicomachus also refers, but less transparently to the fact that rectangular numbers are the mean proportionals between consecutive squares,<sup>108</sup> and observes that squares have only one geometric mean, while cubes have two.<sup>109</sup> Indeed, he also mentions the characterization of the arithmetic mean by square differences and the inequality  $(a + d)^2 > a(a + 2d)$  given by Ibn Sīnā.<sup>110</sup>

Nicomachus then devotes chapter 26 to a digression on the harmonic (composition) proportion in music. In chapter 29, he gives the same recipe for the harmonic mean as Ibn Sīnā does, but neither offers formulas for the smallest term, given the largest and the mean, nor for the largest, given the

<sup>103</sup> See, for example, Rosen (87), 91 ff.

<sup>104</sup> D'Ooge (83), 234-235.

<sup>105</sup> *Ibid.*, 236.

<sup>106</sup> *Ibid.*, 266 ff.

<sup>107</sup> *Ibid.*, 270.

<sup>108</sup> *Ibid.*, 272.

<sup>109</sup> *Ibid.*, 273.

<sup>110</sup> *Ibid.*, 279.

mean and the smallest. Finally, the error in Ibn Sīnā's text:  $mb = 2ma$ , has an analogue-though not an exact one-in Nicomachus'.<sup>111</sup>

Now, let us summarize the most striking differences between Ibn Sīnā's work and that of his apparent model Nicomachus' *Arithmetic*, and try to account for them. It is evident that we should look first, for such variations in Thābit's *K. al-mudhal*.

These differences are primarily interpolations by Ibn Sīnā, and consist of: 1) the formula  $1 + 2 + \dots + n + (n - 1) + (n - 2) + \dots + 1 = n^2$ ; 2) formulas like  $1 + \dots + 6 = 21 = 3(6) + \frac{1}{2}(6)$ , etc; 3) criteria for abundant and deficient numbers; 4) a geometric justification for composition of ratios; 5) formulas for any term in a harmonic proportion, given the other two. Curiously enough, they do not appear in Thābit's *al-mudhal*, which, at least in the edition of Kutsch available to us, follows the Greek text of Nicomachus' *Arithmetic* quite closely, without additions or commentary.

Nonetheless, some of these items do appear in other Greek arithmetics: Iamblichus, for example, in his *On Nicomachus' Introduction to Arithmetic*, discusses the Pythagorean formula

$$\begin{aligned} 1 + 2 + 3 + \dots + (n - 1) \\ + n = n^2, \\ + 1 + 2 + 3 + \dots + (n - 1) \end{aligned}$$

explaining how the Pythagoreans looked upon the sum,  $n^2$ , as a "race-course." ( $\delta\acute{\iota}\alpha\nu\lambda\omicron\varsigma$ ). Iamblichus even cites the same particular instance,  $n=10$ <sup>112</sup> It doubtless appears in other manuals and handbooks of mathematics-perhaps some of them Arabic. On the other hand, formulas 2) may be original with Ibn Sīnā. The faulty criteria for abundant and deficient numbers seem, however, to have no counterpart in Greek literature. Thābit, of course, is known to have left criteria in *Maqala fī stiḥrāg al-a'dād al-mutaḥābba*

<sup>111</sup> *Ibid.*, 282, n.2.

<sup>112</sup> Heath (44), I, 114. However, as Zeuthen points out ("Sur l'arithmétique géométrique des Grecs et des Indiens," *Bibliotheca mathematica* (ser.3) 5 (1904), Nicomachus' statement that asquare is composed of two consecutive triangular numbers (*Arithmetic*, Book II, chapter 12) is equivalent, since this assertion is just  $(1+2+\dots+n)+(12+\dots+n-1)=n^2$ .

bi-suhūlat al-maslak ilā dālika.<sup>113</sup> He states, for example, that if  $s = 1 + 2 + 2^2 + \dots + 2^n$ , then  $2^n s$  is perfect, if  $s$  is prime, while  $2^n p$  is deficient if  $p$  is a prime less than  $s$ , and abundant if  $p$  is a prime exceeding  $s$ . These are proven, and followed by additional criteria for abundant and deficient numbers.<sup>114</sup> The whole tract, as the title indicates, is concerned with establishing a criterion for “amicable” numbers.

As we remarked earlier, the geometric approach to the justification of composition of ratios is already present in such compendia as al-Khwārizmī’s *Algebra* (section on legacies, for example). This leaves 5) : formulas for one term in a harmonic proportion, given the other two. The geometrical versions of these are certainly preserved in Greek sources, e. g., Pappus’ Collection III, propositions 9-11, which give an explicit construction of one term (as a line segment) in terms of the other two.<sup>115</sup> On the other hand, none of the Greek Arithmetics available to us—Pappus, Nicomachus, Theon of Smyrna, Iamblichus—contain a rhetorical solution of the problem. We comment on this again, momentarily.

At this point, to complete our picture of Ibn Sīnā’s work in arithmetic, we offer a sketch of those parts of the *Shifā’* which supplement the arithmetic of the ‘Alā’i and Najāt. These are : a) a short account of “practical” arithmetic, e. g., multiplication tables to  $10 \times 10$  (figure 5) arranged very much like those in Boethius’ manual of arithmetic;<sup>116</sup> (figure 6) [See figure 5 & 6,];

b) various rules of formal algebra; c) a list of triangular, square, ..., octagonal numbers, which coincides, up to scribal error, with Nicomachus II, chapters VIII-XII;<sup>117</sup> d) a fairly complete discussion of the 10 “proportions” of the Greeks; e) a theorem in the Indian arithmetic about squares and cubes (casting-out nines); f) two rules for summation of square and triangular arrays filled with ordered arrangements of successive odd numbers.

<sup>113</sup> Sezgin (98), V, 270, n. 13. See also F. Woepcke, “Notice sur une théorie ajoutée par Thābit ben Korrah à l’arithmétique spéculative des Grecs,” *J. Asiatique* 20 (1852), 420-429.

<sup>114</sup> See Woepcke cited in 103; also Dickson (28), 5.

<sup>115</sup> Hultsch (51), I, 75 ff.

<sup>116</sup> Ibn Sīnā, *Al-Shifā’*: al-Ḥisāb (101), 39. Compare with the table in Boethius, *Arithmetica Boetii*, Augsburg 1488, reproduced in Smith (103), 26, and given in our text. For an interesting study of multiplication tables (sexagesimal) used by Medieval Islamic astronomers, see King (60)-(61).

<sup>117</sup> D’Ooge (83), especially, 248; al-Ḥisāb (Cairo, 1975), 13.

The first of these occupies the third section of the Cairo edition<sup>118</sup> and needs no further comment. The various formal rules of algebra are all rhetorical and involve factorizations, like :  $A^2 - A = A(A - 1)$ ,  $A^3 - A = (A - 1)(A)(A + 1)$ ,  $A^4 - A = A(A - 1)(A^2 + A + 1)$ ,  $(A + B)^2 = 2AB + B^2 + A^2$ , or properties of exponents, such as  $2^{n+1} - 2^n - 1 = 3(2)^{n-1} - 1$  and  $(9) 2^{n-1} - 2 = 3(2^n) - 1 + 3(2^{n-1}) - 1$ . These are scattered throughout the text, and Lutfy attempts to summarize them following the preface to his edition of al-Ḥisāb.<sup>119</sup>

١٠	٩	٨	٧	٦	٥	٤	٣	٢	١
٦٠	١٨	١٦	١٤	١٢	١٠	٨	٦	٤	٢
٣٠	٢٧	٢٤	٢١	١٨	١٥	١٢	٩	٦	٣
٤٠	٣٦	٣٢	٢٨	٢٤	٢٠	١٦	١٢	٨	٤
٥٠	٤٥	٤٠	٣٥	٣٠	٢٥	٢٠	١٥	١٠	٥
٦٠	٥٤	٤٨	٤٢	٣٦	٣٠	٢٤	١٨	١٢	٦
٧٠	٦٣	٥٦	٤٩	٤٢	٣٥	٢٨	٢١	١٤	٧
٨٠	٧٢	٦٤	٥٦	٤٨	٤٠	٣٢	٢٤	١٦	٨
٩٠	٨١	٧٢	٦٣	٥٤	٤٥	٣٦	٢٧	١٨	٩
١٠٠	٩٠	٨٠	٧٠	٦٠	٥٠	٤٠	٣٠	٢٠	١٠

Figure 5

As we have noted, Ibn Sīnā's account of the various Pythagorean numbers follows Nicomachus and is the content of section 3 of al-Ḥisāb (pp. 53-61).<sup>120</sup> The section on the 10 Greek proportions is best summarized by the table one below:<sup>121</sup> (p. 65-69 in text). This is in the last (4th) section. (Section one is devoted to properties of natural numbers).

<sup>118</sup> al-Ḥisāb (Cairo, 1975), especially 39.

<sup>119</sup> al-Ḥisāb (Cairo, 1975), 11-14.

<sup>120</sup> I. e., in Lutfy's edition.

<sup>121</sup> *Ibid.*, 13-14.

	Teträgona.			Longitudo.				Secūda vnitas.			
Prima vnitas.	1	2	3	4	5	6	7	8	9	10	
	2	*	4	6	8	10	12	14	16	18	20
	3	6	*	9	12	15	18	21	24	27	30
	4	8	12	*	16	20	24	28	32	36	40
	5	10	15	20	*	25	30	35	40	45	50
	6	12	18	24	30	*	36	42	48	54	60
	7	14	21	28	35	42	*	49	55	63	70
	8	16	24	32	40	48	56	*	64	72	80
	9	18	27	36	45	54	63	72	*	81	90
	10	20	30	40	50	60	70	80	90	*	100
	Secūda vnitas.			Longitudo.				Teträgona.			

Ratio atq; expositio digeste formis  
 Cap. 27.  
**S**igif duo pma late,  
 ra ppositie formule q  
 facit egulu: ab vno  
 ed. 10. et. 10. pceden  
 tis respiciat: et his sub  
 teriores ordies cōpa  
 rent: qui scilicet a. 4. angulum incipi  
 entes: in vigenos terminū ponunt:  
 duplex id est pma species multipli  
 cationis ostenditur: ita vt pma pma

mā sola superet vnitate: vt duo pma  
 secūda scdm binario supuadat: vt q  
 ternariū binariū. tertū tertiu tribus:  
 vt senariū ternariū. qrtus qrtū qber  
 narij numero nitate transcendat: vt  
 8 qternariū: et p eadē cūcti sequentiā  
 de minoris pluralitate pretereant.  
 Si vero tertū angulus aspiciat: q ab  
 9. inchois lōgitudine latitudine q  
 tricenis alia secus numerū extēdit:  
 et hic cū pma latitudine et lōgitudi  
 ne cōparetur: triplex species multi

MULTIPLICATION TABLE, 1488 BOETHIUS

Figure 6

Table 1. [ $a < b < c$ , positive numbers]

1.  $\frac{a}{a} = \frac{c-b}{b-a}$ ;  $b = \frac{a+c}{2}$ ,
2.  $\frac{c-b}{ab} = \frac{b}{a}$ ;  $b = \sqrt{ac}$ ,
3.  $\frac{c-b}{b-a} = \frac{c}{a}$ ;  $b = \frac{2ac}{a+c}$ ,
4.  $\frac{c-b}{b-a} = \frac{a}{c}$ ;  $b = \frac{a^2+b^2}{a+c}$       example (3, 5, 6)
5.  $\frac{c-b}{b-a} = \frac{a}{b}$ ;  $b = \frac{c-a}{2} + \sqrt{\frac{(c-a)^2}{4} + a^2}$ ,      (2, 4, 5)
6.  $\frac{c-b}{b-a} = \frac{b}{c}$ ;  $b = -\frac{c-a}{2} + \sqrt{\frac{(c-a)^2}{4} + c^2}$ ,      (1, 4, 6)
7.  $\frac{c-a}{b-a} = \frac{c}{a}$ ;  $b = \frac{2ac-a^2}{c}$ ,      (6, 8, 9)
8.  $\frac{c-a}{c-b} = \frac{c}{a}$ ;  $b = \frac{c^2+a^2-ac}{c}$ ,      (6, 7, 9)
9.  $\frac{c-a}{b-a} = \frac{b}{a}$ ;  $b = \frac{a}{2} + \sqrt{\frac{4ac-3a^2}{2}}$ ,      (4, 6, 7)
10.  $\frac{c-b}{b-a} = \frac{a}{b}$ ;  $b = c-a$ ,      (5, 3, 2)

It is worth noting that all the examples, except that for (10) are the same as those given by Nicomachus. (2) and (10) are in error, perhaps owing to mistakes in transcription from the manuscript; the former should read  $\frac{c}{b} = \frac{b}{a}$ , and the latter  $\frac{b}{a} = \frac{c-a}{c-b}$  (or its inversion), with example (3, 5, 8). Finally we call attention to the fact that (4) – (6) are stated in the form given by Theon of Smyrna rather than that of Nicomachus, i.e., as “inversions.”<sup>122</sup>

<sup>122</sup> See D’Ooge (83), 281-284 for tables giving respectively the Nicomachus and the Theon forms. Note, however, that D’Ooge reverses our  $a$  and  $c$  making  $a > b > c$ . Some comments on Theon’s *Mathematics Useful* ... are in order. His treatment of ratio and proportion are relegated to the section of music; the section on arithmetic has the following order :

1. The order in which mathematics should be studied.
2. One and the monad
3. Even and odd numbers
4. Prime numbers

In the section of al-Ḥisab devoted to the Indian arithmetic, Ibn Sīnā, without elaboration gives two rules. These are : 1) “If a given number, which is divided by 9 gives a remainder of 1 or 8, then the square of that number, on division by 9, gives a remainder 1. If the number, divided by 9, gives remainder 2 or 7, then the square of that number, on division by 9, gives remainder 4. If the number, divided by 9, leaves a remainder of 4 or 5, then its square, divided by 9, gives a remainder of 7. Finally, if the number, divided by 9, gives a remainder of 3, 6 or 0, then its square, on division by 9, leaves remainder zero.” 2) “If a number, divided by 9, leaves a remainder of 1, 4 or 7, then its cube, on division by 9, gives a remainder of 1; if the number, divided by 9, gives a remainder of 2, 5 or 8, then its cube, divided by 9, leaves 8; and if the number, divided by 9, leaves remainders 3, 6, or 0, then its cube, on division by 9, leaves remainder zero.” Woepcke<sup>123</sup> appears to have been the first to call attention to these rules, having found them in a Leiden manuscript.<sup>124</sup> Cantor<sup>125</sup> cites Maximus Planudes as offering support for Ibn Sīnā’s own claim that the rules are Indian in origin, while Tannery argues that “proofs by 9” may very well have evolved independently from

5. Composite numbers
6. The various kinds of even numbers
7. Square, unequilateral and parallelogramic numbers
8. Promecic numbers
9. Triangular numbers, the way they are obtained and the polygonal numbers.
10. Pyramidal numbers, lateral and diagonal numbers
11. Perfect, abundant and deficient numbers

Chapter 11 amply illustrates the quality of the work. The translation is Lawlor’s : “Furthermore, among numbers some are called perfect, others abundant, and others deficient. Those are called perfect which are equal to (the sum of) their aliquot parts, such as 6. The parts of 6 are in fact its half, 3, its third, 2, and its sixth, 1, which when added together make 6.

“This is how the perfect numbers are created : If we arrange the numbers in a progression by doubles, starting from unity, and we add them until we obtain a prime, incomposite number, and if we multiply this sum by the last term added, the product will be a perfect number. Let us then arrange the numbers in a progression by doubles, 1, 2, 4, 8, 16. Let us add 1 and 2; the sum is 3; if we multiply this by the last number added, which is 2, we will have 6 which is the first perfect number (since  $1 + 2 + 3 = 6$ ). If we now add the three successive doubles 1, 2, and 4, the sum is 7 ..., 28 perfect.” (The section concludes with a cosmological discourse on the tetraktes!)

<sup>123</sup> Woepcke (122), 502-504.

<sup>124</sup> Cantor (18), I, 756.

<sup>125</sup> *Ibid.*, 756-757.

the Apollonian notion of “pythmene.”<sup>126</sup> However, this may be, the notion of “proof by 9” is certainly in evidence in Arabic arithmetics before the time of Ibn Sīnā.<sup>127</sup> On the other hand, it is worth pointing out that Ibn Sīnā offers his rules without application, and that, although they contain the germ on an idea for a “proof by casting 9’s,” they are merely statements about congruence classes of squares and cubes modulo nine.

Of much more interest are Ibn Sīnā’s two rules for summations of square and triangular arrays of odd numbers. These appear in the first section of al-Hisāb. The first reads: “If the successive odd numbers are placed in a square table, the sum of the numbers lying on the diagonal will be equal to the cube of the side; the sum of the numbers filling the square will be the fourth power of the side.”<sup>128</sup>

Ibn Sīnā illustrates the rule by first giving a 5x5 square :

9	7	5	3	1
19	17	15	13	11
29	27	25	23	21
39	37	35	33	31
49	47	45	43	41

Figure 7

whose diagonal sums  $9 + 17 + 25 + 33 + 41 = 125 = 5^3 = 1 + 13 + 25 + 37 + 49$ , and whose total sum is  $625 = 5^4$ . In addition, he illustrates the way the squares are filled by using 2x2 and 3x3 examples:<sup>129</sup>

<sup>126</sup> Tannery (110), 142.

<sup>127</sup> Saidan (92), and his article “Al-Uqlīdisī,” DSB XIII, 544.

<sup>128</sup> Cairo ed., 25.

<sup>129</sup> *Ibid.*, 25-26.

3	1
7	5

5	3	1
11	9	7
17	15	13

Figure 8

Later in the same section,<sup>130</sup> he discuss square arrays of the ‘doubles,’ giving as an example:

22	18	14	10	6	2
46	42	38	34	30	26
70	66	62	58	54	50
94	90	86	82	78	74
118	114	110	106	102	98
142	138	134	130	126	122

Figure 9

He notes that the sum of each diagonal is equal to the other.

The second rule, which follows immediately after the first is :<sup>131</sup> “If the successive odd numbers are placed in a triangle, the sum of the numbers taken from one row equals the cube of the [row] number.” Ibn Sīnā illustrates this with one figure :

<sup>130</sup> *Ibid.*, 30.

<sup>131</sup> *Ibid.*, 26.

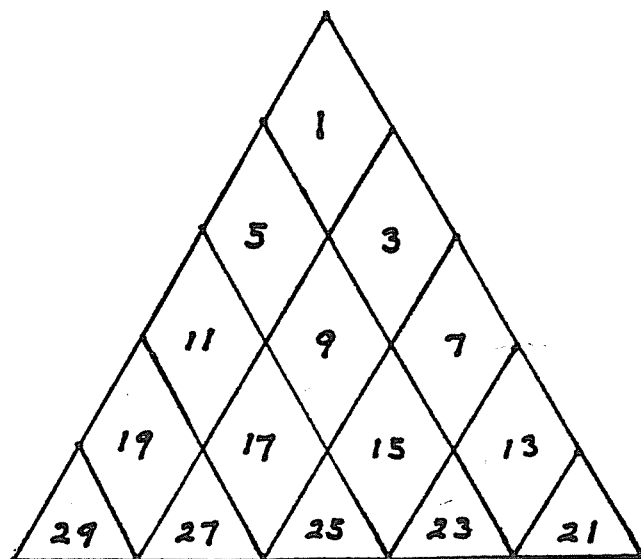


Figure 10

As the reader may verify  $1 = 1^2$ ,  $3 + 5 = 8 = 2^2$ , ...,  $21 + 23 + 25 + 27 + 29 = 125 = 5^2$ , ....<sup>132</sup>

These two rules are quite striking, and for the modern reader, seem like sparkling gems in Ibn Sīnā's otherwise, rather ordinary, but thorough account of the elements of Pythagorean and elementary Indian arithmetic. The second is easily traced; Nicomachus gives it in a rather less eye catching form in the *Arithmetic*.<sup>133</sup> "For when the successive odd numbers are set forth indefinitely beginning with 1, observe this : The first one makes the potential cube; the next two added together, the second; the next three, the third..." The first is a different matter, however. Since our inspection of Arabic manuals of arithmetic like that of al-Uqlīdisī (952 /3),<sup>134</sup> or the later summary of the Hindu arithmetic of Khūshar Ibn Labbān (1000),<sup>135</sup> fail to turn up anything like it, we can only speculate on Ibn Sīnā's source. Perhaps

<sup>132</sup> In their interesting paper on this material from al-Ḥisāb, Kahharov and Hodžiev (55), 122, point out that from explicit formulae for the line sums with  $n = 2k$  and  $2k+1$ , it also follows that "the sum of the numbers situated in successive lines, beginning with the first equals the square of the triangular number which corresponds to the number of the following line. For example, the sum of the numbers filling lines 1 and 2 is  $1 + 3 + 5 = 9 = 3^2$ , of 1, 2 and 3 is  $1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2$

<sup>133</sup> D'Ooge (83), 263-264.

<sup>134</sup> Saidan (92), *passim*.

<sup>135</sup> Kitāb fī usūl ḥisāb al-hīnd; see tr. of M. Levey and M. Petruck (63).

it was the same as that for his rule of nines, an Indian method, or, perhaps, it was his very discovery.

Finally, let us return to our consideration of possible sources for Ibn Sīnā's computation of one term of the harmonic proportion in terms of the other two. As our table of the 10 Greek proportions found in al-Ḥisab indicates, Ibn Sīnā also solved for the middle term as a function of the other two, indicating familiarity with a (rhetorical) quadratic formula. Perhaps, in the case of the 'Alā'ī as well as the Shifā', this work is an exercise the author introduces to show his familiarity with methods for solving linear and quadratic equations.

Curiously enough, the author himself is rather misleading about his actual sources. As a matter of fact, in the introduction to the arithmetic section,<sup>136</sup> he says that he relies on the Kitāb al-Uṣṭuqūsāt for his number theory, and Kitāb al-Katêguryas for his basic ideas about numbers. The former is, of course, Euclid's Elements,<sup>137</sup> and the latter is probably Aristotle's Categories (which does contain some primitive ideas about numbers), or some commentary on it.<sup>138</sup>

#### IV. Geometry

The geometric treatise of Ibn Sīnā in the 'Alā'ī is, as al-Jūzajani tells us in his introduction, a resumé of the *Elements* of Euclid, in particular the parts necessary for the study of the *Syntaxis Mathematica* (Almagest) of Ptolemy.<sup>139</sup> It consists of 13 sections, does not list definitions, postulates and axioms, and contains no formal proofs. In what follows we offer a brief summary of these sections, test the statement that they offer sufficient geometric preparation for the *Almagest*, and examine the possibility of a more immediate source than the *Elements*.

<sup>136</sup> Cairo ed., 17.

<sup>137</sup> See for example, Sezgin VI, p 103 where one is led to speculate on the origin of the title al-Uṣṭuqūsāt. Is it a corruption of the Greek στοιχειών, or from the Arabic uṣṭuqsa? The second author wishes to thank A. S. Ibrahim and Simon Samoeil of the University of Petroleum and Minerals for some interesting conversations on this point.

<sup>138</sup> Perhaps he is referring to al-Fārābī's, or even his own.

<sup>139</sup> Alā'ī (100), 91.

For the reader's convenience we list those propositions of Euclid used implicitly in the *Almagest* (Table 2.)

Table 2. Euclidean Propositions Implicit in the *Almagest*

Location in <i>Almagest</i>	Proposition of <i>Elements</i>
Book I, section 10	I, 11, 10; II, 6; I, 47; VI (def. 3) XIII, 9; IV, 15 (Cor.); XIII, 10 III, 21; VI, 4, 16; II, 1; III, 27; I, 26 VI, 8; III, 26, 29; VI, 3; V, 8; VI, 1 III, 3; I, 21; VI, 33
Book I, section 13	VI, 4, 2
Book III, section 3	I, 16; III, 7, 3; III, 16, Por; VI, 8 VI, 7
Book III, section 5	III, 31
Book IV, section 6	III, 36
Book IX, section 10	I, 32. <sup>140</sup>

In table 3 below is a list of the Euclidean propositions in order by book. This allows a verification of the fact that only the first 6 books of the *Elements* play an essential role in the *Almagest* (with the exception of a proposition from Book VIII and another from Book XIII).

Table 3.

I (Book of <i>Elements</i> ) Proposition	10, 11, 16, 21, 26, 32, 47
II	1,6
III	3, 7, 16 Por, 21, 26, 27, 31, 36
IV	15 (Cor.)
V	8
VI	(def. 3), 1, 2, 3, 4, 7, 8, 16, 33
VIII	9
XIII	10

<sup>140</sup> See *Syntaxis Mathematica*, ed. J. L. Heiberg. Teubner, 1899-1907.

The first three sections of Ibn Sīnā's geometry are concerned with the properties of intersecting straight lines, parallel lines, and notions related to angles and triangles. Ibn Sīnā begins the first section by observing that two lines intersect at right angles or obliquely, and in either case the sum of the angles on the same side is two right angles (*Elements* I, 13).<sup>141</sup> In addition the opposite angles are equal (I, 15). His definition of perpendicular lines: one "does not incline toward one side or the other of the second line", stands in contrast to Euclid's definition (I, definition 10): "making equal adjacent angles." Lines and planes are undefined. Ibn Sīnā then closes this section with the following propositions : A line perpendicular to two intersecting lines is perpendicular to every line in their plane and passing through the intersection point (XI, 2, 4). He adds that such a line is said to be perpendicular to the plane (XI, def. 3), and states that lines perpendicular to the same plane are parallel (XI, 6), offering a heuristic argument based on the notion of "inclination."

In section two, the author notes that line segments, when extended, either intersect, or are parallel. They intersect when one is "inclined toward the other," the intersection being along the direction of inclination, but not in the opposite. Moreover, "if they always remain separated in such a way that the distance between the 'extremities' are equal, and the perpendicular to one is also perpendicular to the other... then they are parallel." This is essentially Posidonius' definition of parallels, which entered the Arabic tradition through Al-Nayrīzī's commentaries on Simplicius' Euclid, and lies within the "equidistance", "convergence-divergence" theory of parallels which extends from Thābit through al-Ṭūsī.<sup>142</sup> Ibn Sīnā then continues with a circular proposition: "each time that a line cuts two parallels without inclining toward one or the other, it is perpendicular to both." Next, he makes an assertion equivalent to *Elements* I, 29, and follows this with : "if a line cuts two other lines so that the two interior angles on the same side equal two right angles, then the two lines are parallel. (I, 28). He argues that if the two lines were not, then all three lines would form a triangle with

<sup>141</sup> Proposition numbers follow the edition of Heath (46).

<sup>142</sup> See the forthcoming paper of al-Daffa' and Strayls, "Naṣir al-Dīn al-Ṭūsī's Attempt to Prove the Parallel Postulate of Euclid," Proc. Int'l Symposium on History of Science at Rāwalpindi, Dec., 1979. This work surveys the entire Islamic-Arabic contribution to postulate V.

the sum of its angles being greater than two right angles, and having an exterior angle equal to one of its interior angles. Of course, the argument implicitly assumes Elements I, 32. to reach a contradiction. Ibn Sīnā then adds that if the sum of the two angles is less than two right angles, the lines actually intersect. He follows this with a construction of a line through a given point and parallel to a given line (I, 31). However, in contrast to Euclid, Ibn Sīnā uses *Element I, 11* implicitly in his construction instead of I, 23, and evidently replaces Euclid's use of I, 27 with I, 28. The second section closes with a statement of I, 30 : lines parallel to the same line are parallel to each other.

He begins section three with a partial statement of I, 32 : the sum of the angles of a triangle is two right angles. The proof offered is accompanied by a diagram (Figure 11):

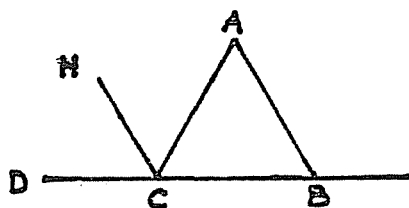


Figure 11

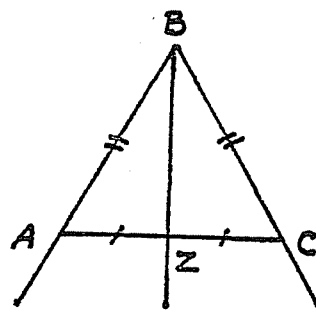


Figure 12

The angle HCD is constructed equal to  $\angle ABC$  to make HC parallel to AB, making implicit use of I, 23 instead of simply citing the previous section. The proof then proceeds as the diagram suggests by reference to Ibn Sīnā's version of I, 29. Next, he enunciates one case of congruent triangles, namely that of equal angles included between two equal sides (I, 4); he later makes implicit reference to three sides equal (I, 8), but omits the case of one side and two angles (I, 26) entirely. His proof of congruence is by superposition, phrased as "apply angle B [in one triangle] to the corresponding angle H by an act of imagination," which seems to indicate an opposition to the introduction of kinematics to geometry, a principal favored by Thābit. Ibn Sīnā next gives a proof of the fact that triangles with two equal sides have two equal angles, with equal supplementary exterior angles (I, 5). It depends on the following figure in which Z bisects AC (I, 10), consequently making

$\triangle ABZ = \triangle BCZ$  (I, 8). (See figure 12 above). This, in combination with implicit use of I, 13 completes his proof. It is, of course, circular since the I, 10 requires I, 9, which in turn depends on I, 8 and I, 8 invokes I, 7 which depends on I, 5 itself. No mention is made of Euclid's partial converse: I, 6, which is considered in totality in al-Nayrīzī's commentary. Ibn Sīnā then observes that if a  $\triangle ABC$  has base  $AB$  increasing, the  $\angle ACB$  will also. This leads him to assert that the greatest side of a triangle is opposite the greatest angle, and conversely (I, 18; I, 19).

In the fourth section, Ibn Sīnā discusses three ruler and compass constructions : perpendicular to a line at a point on the line (I, 11), perpendicular to a line from an external point (I, 12), bisector of a given angle (I, 9). The following section is devoted to parallelograms, triangles and their areas. It begins with a definition of parallelogram, followed by the assertion that a diagonal separates the parallelogram into two identical triangles (I, 34 approximately). Then there appear two propositions : parallelograms (triangles) with the *same* base, lying between parallel lines (i. e., with the same height) are equal in area. The discussion then "generalizes" this to include parallelograms (triangles) with equal bases (I, 35, 36, 37, 38). The final proposition seems to be in reference to a figure that does not appear in the manuscript of 'Alā'ī; it asserts that the parallelograms formed by completing the triangles, on either side of the diagonal of a given parallelogram, to parallelograms, gives rise to equal parallelograms.

In section 6, Ibn Sīnā takes up the "multiplication of lines." First he shows how to "multiply" a line segment by itself, that is, how to construct a square with side equal to the given segment (I, 46). He then offers sketches of proofs of the Pythagorean theorem (I, 47) and its converse (I, 48), following Euclid in spirit. Yet, he makes no mention of Thābit's alternate proof of I, 47 (which is noted in al-Nayzīrī's commentary), and Thābit striking generalization to arbitrary triangles is also omitted.<sup>143</sup> Ibn Sīnā's next proposition in geometric algebra is: if a line segment be divided into two parts, the square of the whole segment equals the sum of the squares on the two subsegments plus twice the product of these two segments.  $[x^2 +$

<sup>143</sup> Sayılı (95), 35-37. See also the forthcoming paper of al-Daffa' and Stroyls. "The Pythagorean Theorem and Mumpsimus," read at the November, 1980 meeting of the AMS.

$(e-x)^2+2(e-x)x=e^2$ ]. This is Elements II, 4 and the treatment again follows Euclid. In an analogous fashion, Ibn Sīnā ends the section with Elements II, 6, which asserts in modern notation that  $(e+m)m+(\frac{e}{2})^2=(\frac{e}{2}+m)^2$ .

Section 7 of the ‘Alā’ī treats circles. Since the reader already has a fair idea of Ibn Sīnā’s method, our account will be very much abbreviated. In order, Ibn Sīnā gives *Elements* III, 26-29 (approximately), that is, propositions concerned with equal circles, chords, central angles and arcs. Then he establishes III, 3, part of III, 18, but skips III, 16-Euclid’s account of the “horned angle,” a topic treated in the manuscript “al-Zawiya.”<sup>144</sup> After that he takes up III, 19, 20, 21, 31 (partially), and 36, with the proof in two parts as it appears in the Elements.

The next three sections (8-10) deal with proportion, with the last of these devoted to geometric figures. The language is clearer and the wording more concise than in the corresponding section of the “Arithmetic” of the ‘Alā’ī. For example, Ibn Sīnā explains the concepts of proportion and composition of ratios straight away, then takes up “inverses,” i. e.,  $\frac{A}{B} = \frac{C}{D}$  then  $\frac{B}{A} = \frac{D}{C}$ . The geometric material consists of 11 propositions from Elements VI : 1, 2, 4 (implicitly), 5, 8, 13 (construction of mean proportional), 11 (third proportional), 14 (reciprocally proportional parallelograms), 16, 19, 20. These, for the most part relate ratios of areas of similar polygons to ratios of squares of lengths of corresponding sides.

In sections 11 and 12, Ibn Sīnā again returns to the topic of circles. Section 11 actually begins by repeating certain parts of section 7, then sketches proofs of *Elements* XII, 1 and 2, while section 12 is concerned with chords of circles, and begins with the construction of a regular hexagon (IV, 15). However Ibn Sīnā’s concern is not with the construction itself but rather the relationship between the side of the hexagon and the diameter of the circle ( $s_6 = r$ ). This is borne out by the fact that he dispenses with the construction of an equilateral triangle altogether, and just assumes it as given (inscribed in a circle), when arguing that  $s_3^2 = r \cdot 3r$ . Then he discusses Elements XIII, 9 which asserts that the side of a hexagon inscribed in a circle is the mean proportional between the side of a decagon inscribed in the same circle and

<sup>144</sup> See section below on the “Foundations”.

sum of the sides of the sides of the hexagon and decagon, and Elements XIII, 10 ( $s_5^2 = s_6^2 + s_{10}^2$ ).

The last section concerns itself with some notions of solid geometry and the properties of spheres. Solids are defined in the Euclidean manner, and spheres, cubes and prisms are given as examples. Then Ibn Sīnā remarks that solids will be considered to be equal if their extremities (surfaces) are equal; he follows this with definitions of cones and prisms. The former includes Euclid's separate class of pyramids, while the latter is more restrictive than the Euclidean class, admitting only prisms with triangular bases and tops, and sides that are parallelograms. This set of definitions is succeeded by a loose discussion of the content of XI, 25, 29-31. After this, Ibn Sīnā devotes considerable attention to pyramids with triangular base, starting with Euclid's dissection (XII, 3) of such a pyramid into two similar, equal pyramids and two equal prisms, each greater than half the original pyramid. Ibn Sīnā's restriction of the class of prisms causes no difficulty, since those referred to in Euclid are of the same type. He follows this with expositions of XII, 4 and 5, but omits Euclid's extension of XII, 5 to pyramids with polygonal bases, preferring instead to illustrate XII, 7: the decomposition of a triangular based prism into three equal triangular based pyramids. Next Ibn Sīnā gives *Elements* XII, 16 and 17 : the construction of polygons between distinct concentric circles, not touching the smaller circle, and polyhedral solids between concentric spheres, not touching the smaller sphere. After mentioning that spheres stand to one another in volume as the ratio of the cubes of their diameters (XII, 18), Ibn Sīnā ends his account of geometry with the assurance that anyone who understood it would be prepared to understand the *Almagest* under the guidance of a good teacher.

Table 4 below summarizes the specific Euclidean propositions that appear in the 'Alā'ī geometry by book as uncovered by our examination. A comparison with the needs of the *Almagest* reader indicate that Ibn Sīnā omits I, 16, 21, 26; II, 1; II, 7, 16 Por; VI, 3, 7, 33; VIII, 9. However I, 32 implies I, 16, while III, 16 Por. and VI, 33 are implicit in the 'Alā'ī. So one must indeed.

Table 4

Book of Elements	Propositions
I	4, 5, 8, 9, 10, 11, 12, 13, 15, 18, 19, 23, 28, 29, 30, 31, 32, 34, 35, 36, 37, 38, 46, 47, 48
II	4, 6.
III	3, 18, 19, 20, 21, 26, 27, 28, 29, 31, 36
IV	15
VI	1, 2, 4, 5, 8, 11, 13, 14, 16, 19, 20
XI	2, 4, dfn. 3, 6, 25, 29, 30, 31
XII	1, 2, 3, 4, 5, 7, 16, 17
XIII	9, 10

admit that Ibn Sīnā provides his reader with a good summary of geometric prerequisite for the *Almagest*.<sup>145</sup>

If Ibn Sīnā used an existing summary (of the *Elements*) by another author, as he did when writing of arithmetic, it may never be known. In fact, it would even be a considerable achievement to determine whether Ibn Sīnā worked with a copy of the *Elements* based on the translation of al-Ḥajjāj Ibn Yūsuf Ibn Matar (fl. 786-833), or the later translation by Ishāq, improved by Thābit Ibn Qurra. The present writers are of the opinion that the summary is probably Ibn Sīnā's own, perhaps based on the translation of al-Ḥajjāj. He certainly would have had to prepare one in his initial, almost single handed study of the *Almagest*, and prior to that he says, in the autobiographical fragment, that he studied Euclid, making no

<sup>145</sup> One might profitably compare this with al-Battānī's introduction to the *Almagest* (Nallino (80), 6 ff.) which starts with an account of the sexagesimal arithmetic. Also, as the reader may discover for himself from perusing the appendix (K. al-Shifā'-Geometry), all the propositions that Ibn Sīnā omits in the *Alā'i*, with the exception of III, 16 (por), and VI, 7 do appear in the *Shifā'*. The omission of VI, 7 is itself a curiosity, since it appears in the "Arabian Euclids". See Hāt (46), I. 80-81. However the omission of the porisms is a common feature of all Arabic editions. Of course, there is also the fact that Ibn Sīnā includes some 40 propositions not essential to the *Almagest*. On the whole, however, his inclusion of these propositions lends some continuity to his account of geometry, and gives us some feeling for his own geometric preference, viz., problems of dissection and construction.

mention of summaries, as he specifically does when telling how a gloss, or summary of Aristotle by al-Fārābī helped him to understand the intent of the original. Moreover, as al-Jūzajānī tells us, Ibn Sīnā revised at least one of his geometrical summaries to include additional figures, a further suggestion that a manuscript of the *Elements* may have been the primary source. A careful examination of the figures themselves provides little evidence for identification of the source, since they differ from both the al-Ḥajjāj and Ishāq-Thābit versions. It is also known that Abū ‘Uthman al-Dimashqī (fl. ca. 908-932) translated parts of the *Elements*, but only the part dealing with Pappus commentary on book X survives, a fact which does not help in deciding the tradition for the ‘Alā’ī text, since it does not contain propositions from the tenth book of the *Elements*<sup>146</sup>.

As Murdoch points out, summaries (mukhtaṣar) were not unknown, even in the Arabic, by Ibn Sīnā’s time. In fact Ḥajjāj favored his second version of the *Elements* for its shorter length<sup>147</sup>. Moreover, those who would argue for a more immediate source of Ibn Sīnā’s summary than any translation of the *Elements*, will certainly point out that Ibn Sīnā, when preparing the *Shifā’*, *Najāt*, and ‘Alā’ī was writing in a well established classical tradition : the encyclopedia format, which has roots in the Greek and Latin worlds of the second century A. D.<sup>148</sup> One of the earliest of these encyclopedias, Theon of Smyrna’s *περὶ τῶν μαθηματικῶν χρησιμῶν εἰς τὴν Πλάτωνος ἀνάγνωσιν* (On the Mathematical Knowledge which is Needed to Read Plato) contained five parts : arithmetic, geometry, stereometry, astronomy and music<sup>149</sup>. Only the first part, with an appendix on music (from a Byzantine edition), and the part on astronomy survive in full, although some are of the opinion that Theon simply never fulfilled the promise of his introduction, to write detailed sections on geometry and stereometry<sup>150</sup>. Instead he limited his remarks in these areas to “geometrical numbers”, e.g., pyramidal numbers, spherical numbers, etc. Heath, in fact, gives some striking evidence to support this opinion<sup>151</sup>. There is, of course, no evidence

<sup>146</sup> Murdoch (78), 439.

<sup>147</sup> Murdoch (78), 439.

<sup>148</sup> Stock (107), 7-8. See also Collison (23) for more details on this tradition, especially 21-81.

<sup>149</sup> Huxley (52), 325-6; Heath (44), I, 112-113; Sarton (94), I, 272.

<sup>150</sup> Sarton (94), I, 272.

<sup>151</sup> Heath (44), I, 112-113.

that Theon's "vade-mecum for philosophy student"<sup>152</sup> entered the Arabic tradition. The same seems to be true of the work of Iamblichus of Chalcis [in Syria] (ca. 250-330) who wrote an encyclopedia of Pythagoreanism, which contained a summary of Nicomachus' *Arithmetic*, as well as books on gematria, geometry and music<sup>153</sup>.

On the other hand, various compendia of the sciences in Arabic, such as the *Mafātiḥ al-'ulūm* (Keys of the Sciences) of Abū 'Abd Allah Muḥammad Ibn Aḥmad Ibn Yūsuf Al-Khuwārizmī (fl. ca. 975 in Khuwarizm)<sup>154</sup>, may well have been known to Ibn Sīnā. The *Mafātiḥ* relies on Nicomachus as well as on the encyclopedia (*Rasā'il*, literally "epistles") of the Ikhwan al-Ṣafā', but appears to be more a catalogue of sciences and terms, with emphasis on their etymologies, rather than a purposeful summary of the fundamentals of each science. Wiedemann devoted his *Beiträge XIV* to a translation of its mathematical sections, and mentions therein other tracts in the scientific encyclopedia tradition in Arabic<sup>155</sup>. The *Mafātiḥ* itself seems to have been written as a manual to assist the reader in partaking of scholarly conversations, or at least to allow him to glean something from listening to them. The author dedicates his work to the vizier of the Samanid ruler Nūḥ II (976-997), Abu'l-Hasan al-Utbī, much as Ibn Sīnā dedicated his *Dānesh-nāmeḥ* to 'Alā' al-Dawla. Yet none of these compendia mentioned, nor those others studied by Wiedemann have any claim for being a source for Ibn Sīnā's geometry, coming between him and Euclid.

Now let us consider Ibn Sīnā's much fuller treatment of the *Elements*, the geometry section of the *Shifā'*. This has recently been edited (in Arabic) by Sabra<sup>156</sup>, and it is his edition to which we shall refer. Unlike the geometry of the *Alā'ī*, this work is a summary of the *Elements* I-XIII, as well as of the post-Euclidean Books XIV-XV. It has no general introduction, preface, or statement of purpose, but each book contains some of the customary

<sup>152</sup> Huxley (52), 325.

<sup>153</sup> Lloyd (66), 1; Heath (44), I, 113-117.

<sup>154</sup> Vernet (118), 357-358.

<sup>155</sup> Wiedemann (120), I, 400-463.

<sup>156</sup> Sabra (102).

Euclidean definitions. The first also enumerates the common notions ('ilm gami'), but omits the postulates, except for the fifth<sup>157</sup>.

Ibn Sīnā's treatment of the Euclidean propositions roughly resembles the *construction* which *follows* each proposition in the *Elements*. In other words, Ibn Sīnā rarely states the proposition itself, but allows it to be inferred from the geometric construction, which acts, in turn, as a guide to the proof. Moreover, he follows Euclid's lead in rarely citing propositions upon which the proof depends, rather than emulating Ḥajjāj, who sprinkled his account of the *Elements* with phrases like "by proposition so and so", etc.<sup>158</sup>.

The two tables below are extracted from Appendix I of this paper and together summarize the contents of Ibn Sīnā's geometry, and the order of the propositions. These are extremely important pieces of information for determining the Arabic edition of the *Elements* used in the *Shifā'*.

TABLE 5

Book	Contents (number of propositions)	
	Euclid	Ibn Sīnā
1	48	53 (but # 42 is missing), accounting for 46 Euclidean propositions, and omitting, I, 40, 45. [53 (# 42 missing); 46 E; omits I 40, 45.]
2	14	18 (but several numerals repeated or added by editor); 14 E; omits none.
3	37	36 (#24 is missing and might correspond to III 26); 35 E; omits III, 26, 29.
4	16	18; 16 E; omits none
5	25	25; 22 E; omits V 21, 22, 23
6	33	31 (also one un-numbered); 27 E; omits VI 7, 9, 12, 16, 27,
7	39	41; 39 E; omits none
8	27	25; 24 E; omits VIII 11, 17, 20
9	36	36; 36 E; omits none

<sup>157</sup> However, postulate 3 of *Elements* I is interpolated between Ibn Sīnā's treatment of I, 2 and I, 3.

<sup>158</sup> See Health (46), I, 83.

10	115	108; 100 E; omits X 7, 8, 10, 16, 19, 27, 29, 30, 31, 32, 83, 84, 112, 113, 114
11	39	39; 37 E; omits XI 23, 34
12	18	16; 15 E; omits XII 6, 13, 14
13	18	16; 15 E; omits XII 6, 13, 14
	(465)	(463) (425E)
Book	Euclid	Ibn Sīnā
14	(9)*	11; 6 *(See Appendix and Heath (46), III, 512-19)
15	(5)**	5; 5 *(see Appendix, and Heiberg and Menge (48), V, 40-67)
	479	479 436

TABLE 6

Order of Propositions in the *Shifā'*

*Shifā'*-Book

1. Same as Elements I (Heath edition throughout)
2. Same as Elements II
3. Same as Elements III
4. Same as Elements IV
5. Interchanges V, 12, 13
6. VI, 1-8, 13, 11, 14, 15, 10, 17, 19, 20, 18, 21, 22, 24, 26, 25, 23, 28, 30, 32, 31, 33
7. VII, 1-20, 22, 21, 23-28, 31, 32, 29, 30, 33-39
8. Same as Elements VIII
9. IX (27, 25, 28)
10. X (15, 11, 17, 14, 18); (21, 13, 22); (74, 76, 75, 77)
11. Same as Elements XI
12. XII (7, 9, 8, 10, 12, 11, 15).
13. XIII (6, 12, 9, 10, 8, 11, 13, 15, 14, 16)
14. Same as Hypsicles<sup>159</sup>
15. (3, 2, 1, 4, 5)<sup>160</sup>

<sup>159</sup> Heath (46), III, 512-19.

<sup>160</sup> See reference to Book 15 in table I.

Curiously enough, although Ibn Sīnā does omit certain propositions from his summary of the Elements, there are only two additions: Book 11, 32 and 33, which are introductory to Book 11, 34 (XI, 33). As the reader will immediately observe from table 1, a mere count of propositions tells us very little about the actual content of the Shifā's geometry. For example, the Heath-Euclid and the Shifā' contain 479 propositions, but, in the latter case, these actually collapse to 436 Euclidean propositions, the remainder being accounted for by errors in numbering, case-wise treatment, or addition of propositions.

As Sabra notes in his introduction<sup>161</sup>, it is very difficult, linguistically to decide whether Ibn Sīnā based the Shifā' on the al-Ḥajjāj or the Ishāq-Thābit edition of the Elements. He does point out, for example, that both Ibn Sīnā and al-Ḥajjāj (2nd version) use 'ilm gami' for "common notion", while Thābit prefers 'ilm'am mutafq alaih. Of course this is far from conclusive, and Sabra goes on to observe that the figures in the Shifā' differ from both those in al-Ḥajjāj and Thābit.

If we assume either a Ḥajjāj or Ishāq-Thābit precursor, at least three types of evidence seem to be available for making a choice : figures, linguistic analysis, order of propositions, including omissions and additions. In the case of al-Ḥajjāj, none of these are entirely compelling since his translation is really an integral part of al-Nayrizi's commentary, and may, consequently, have been extensively re-worked. Moreover, any judgment based on the geometric figures tends to be extremely subjective. For example, one of the authors of this paper compared the figures in Book I of al-Ḥajjāj with those in Ibn Sīnā, and felt that 16 figures agree, 23 show partial agreement and 7 show marked discrepancy. In fact, al-Jūzajānī seems to imply, in the biography, that Ibn Sīnā made some entirely new figures for the *Elements*.

Since the present writers had only Book 1, 5-6 of Codex Leidensis 399, 1 (al-Ḥajjāj) and fragments of Nicoll 279 (Ishāq-Thābit) at their disposal while preparing this paper, no additional linguistic analysis of the three versions of Euclid is offered here. However, to give the reader some idea of the complexity of the problem in this area, we offer some excerpts (V, 7, 15) from Ibn Sīnā (Sabra's edition), al-Ḥajjāj (Codex Leidensis) and Ishāq-

<sup>161</sup> Sabra (102), 11.

Thābit (Copenhagen 81, studied by Klamroth), in Appendix II. As the reader will note, from the short, though typical selection, the principal difficulty is caused by the brevity of Ibn Sīnā's summaries. We hope to return to this intriguing identification problem, from the linguistic point of view, in a later paper. However, we can contribute something by turning to the third type of evidence: the ordering of propositions. From this point of view, the evidence seems to be slightly in favor of an al-Hajjāj precursor over Ishāq-Thābit. Let us review this evidence, from Appendix I, by book. Again, the evidence is not decisive, because there is really not much variation in order, omission and addition.

In book one al-Hajjāj, Ishāq-Thābit and Ibn Sīnā follow the Euclidean order. The first gives 47 propositions, the second 48, and the last 46. Both al-Hajjāj and Ibn Sīnā however omit proposition 45. In fact, Ibn Sīnā also omits proposition 40, both in the Sabra edition, and in the Leiden manuscript studied by Lokotsch. In book two, all three editions are identical with each other, in order, and indeed agree with Euclid. Nonetheless, Ibn Sīnā seems to lean slightly toward al-Hajjāj by giving II, 14 for triangles, instead of arbitrary rectilinear figures.

In book three, all versions are in the Euclidean order, with Ibn Sīnā only, omitting III, 26, 29. He also agrees with both al-Hajjāj and Ishāq-Thābit by treating III, 11 and 12 as a single proposition. The omission of part two of III, 15 which occurs in the Leiden manuscript (Lokotsch) is amended in the Cairo edition. In the fourth book, the only distinguishing characteristic, two different proofs of a single proposition in Ishāq-Thābit is not shared by either al-Hajjāj or Ibn Sīnā.

Book 5 in the Cairo edition differs somewhat from the Leiden manuscript. While the former omits V, 21, 22, 23 (possibly a last leaf ?) and inverts V 12, 13, the latter omits only V, 6 and inverts V, 10, 11. Consequently, the Cairo edition agrees in its ordering with both Hajjāj and Ishāq-Thābit, although the latter editions are both complete.

The peculiar ordering of the last planimetric book, six : 1-8, 13, 11, 12, 9,10,14-17,19, 20,18, 21-22, 24, 26, 23, 25, 27-30, 32, 31, 33 *should* be decisive. Unfortunately, both al-Hajjāj and Ishāq-Thābit share it. Ibn Sīnā, in the Cairo edition presents the propositions as : 1-8, 13,11,14-15,10,17,19-

20, 18, 21-22, 24, 26, 25, 23, 28, 30, 32, 31, 33, omitting 7, 9, 12, 16, 27, 29. The Leiden manuscript (Lokotsch) nearly agrees : 1-8, 13, 9, 14-15, 10, 17, 19-20, 18, 21-22, 24, 26, 23, 25, 27-30, 32, 31, 33, omitting 11-12, 16.

Since the al-Hajjāj edition exists only in six books, with fragments of seven, it is impossible to carry this triple comparison further. In summary, we see that Ibn Sīnā seems to favor al-Hajjāj slightly in books one and two. In the other four books, all editions agree remarkably well.

If one carries this analysis further, comparing only Ishāq-Thābit and Ibn Sīnā (Cairo edition, Leiden ms. has only I-VI), one finds extremely good agreement. For example, both preserve the curious orders of books seven, twelve, thirteen, namely VII, 1-20, 22, 21, 23-28, 31-32, 29, 30, 33-39; XII, 1-5, 7, 9, 8, 10, 12, 11, 15-18; XIII, 1-3, 5, 4, 6, 7, 12, 9, 10, 8, 11, 13, 15, 14, 16-18.<sup>162</sup> Each has Euclidean order in book eight, although Ibn Sīnā omits Ishāq-Thābit non-Euclidean propositions 24-25. Moreover, since Ibn Sīnā omits XI, 33, 34, which are reversed in Ishāq-Thābit, there is again agreement order-wise, although Ibn Sīnā fails to divide XI, 31 into two parts, a feature of Ishāq-Thābit. This leaves only IX and X of the Euclidean books; here there are some discrepancies, as the reader may observe from consulting Appendix I.

## V. Trigonometry

Ibn Sīnā devotes a section of the *Shifā'* (Dzawamī Kitāb Bitlimus al-Kabir al-ma'mul Madjisti wa 'ilm al-haya) to Ptolemy's astronomical treatise, the *Almagest*. A part of this, in turn, contains his work on trigonometry, which is unique to the *Shifā'*, in the sense that it is not summarized in the *Najāt* or in the 'Alā'ī. Since the Cairo edition of this part of the *Shifā'* has not yet appeared, we can only report on its contents and sources by referring to a Russian analysis given by Hairetdinova.<sup>163</sup> That writer has proven that Ibn Sīnā's trigonometric materials follow those of Abu Nasr al-Fārābī's (870-950) *Sharh al-Madjisī* (Commentary on the *Almagest*),<sup>164</sup> almost verbatim.

<sup>162</sup> This is nearly the case, since Ibn Sīnā omits XIII, 4, 5, 7.

<sup>163</sup> Hairetdinova (41).

<sup>164</sup> A Russian edition of approximately one half of al-Fārābī's commentary (*Kommentarii k Almagisti Ptolemeya*) was prepared by A. Kubesov, *et. al.*, Izdat. "Nauka" Kazah. SSR, Alma-Ata, 1975. His supplement to the *Almagest*, al-Lawaḥiq, is included (pp 53-90) of B. A. Rosenfeld's *The*

Both works treat the transversal theorem of Menelaus' (c. 100 A. D.) *Sphaerica*,<sup>165</sup> following Menelaus' own account more closely than the later adaptation by Claudius Ptolemy. They also diverge in some noteworthy aspects from both of these authors; for example, "sine" is defined, possibly (in al-Fārābī's case) for the first time in an Arabic commentary on Ptolemy.<sup>166</sup> it is given as "half the chord of double the arc."<sup>167</sup> in addition, al-Fārābī and Ibn Sīnā present a result equivalent to the planar law of sines for triangles inscribed in a circle, in the form, "If the angles are known then we determine the ratios of their sides. If the angles are inscribed in a circle and the arcs of each angle are known to me, then the [ratio of sides equals] the ratio of the corresponding chords to the diameter of the circle; moreover if the angle is right, then chord and diameter coincide, Therefore, if one of the angles is known, or one of the arcs of a side is known and its ratio with the chord of a right angle, this is sufficient [information] for us to determine the arc of that corresponding angle."<sup>168</sup> They also take special care in treating certain cases of Menelaus' theorem missing from both *Sphaerica* and the *Almagest*, as well as in justifying Ptolemy's handling of composite and "inverse" ratios.<sup>169</sup> Their treatment, in the latter, is strictly Euclidean.

Hairetdinova goes on to show that al-Fārābī's work is quite different from Thābit Ibn Qurra's tract *Risala fi shakl al-kita'* (Treatise on the transversal theorem).<sup>170</sup> In fact, the differences extend even to the terminology: al-Fārābī, for example, uses "nayf" rather than "fadl" for "difference", and "nisba al-murakkaba" rather than "nisba al-mu'allafa" (composite ratio). From this Hairetdinova concludes that al-Fārābī may well have translated the *Sphaerica* directly from the Greek, coining his own terms as he worked.

*Mathematical Treatises of al-Fārābī*, Alma-Ata, 1972. For reviews of each, see MR 57, # 5845 and MR 55, # 7711.

<sup>165</sup> For a statement of the planar version of Menelaus' theorem and its extension to the sphere-along with some good historical perspectives Kennedy (58), 342-345, and Heath (44), II, 261-274.

<sup>166</sup> However, see Kennedy (58), 346 for a 4th or 5th century use of the sine in a Sanskrit astronomical work, *Surya Siddhanta*.

<sup>167</sup> Hairetdinova (41), 30 gives exact reference to the Arabic ms. in note 9.

<sup>168</sup> For example, if a, b, c are sides and A, B, C are opposite angles  $\frac{1}{2} = \frac{a}{2r} / \frac{b}{2r} \sin A / \sin B$ , etc., where r is the radius of the circle.

<sup>169</sup> See the section on arithmetic above

<sup>170</sup> See A. Björnbo, "Thabits werke über den Transversalensatz," *Abh. z. Geschichte d. Naturwiss. u. Med.* Heft VII, 1924.

It is also worth noting that this section of the *Shifā'* received some attention by other Arab astronomers, e. g. al-Nasāwī (11th c.) cites it in his *Kitāb al-ishba' fi shark ash-shakl al-kita'*.<sup>171</sup>

## VI. Foundations

When one recalls the appraisal of a thirteenth century editor of Euclid, probably Muḥyi'l-Din al-Maghrabī,<sup>172</sup> of Ibn Sīnā's work in geometry, one may question the usefulness of looking for statements on the foundations of mathematics in his writings. The editor wrote, rather bluntly, "Ibn Sīnā left out the postulates and many definitions, and had nothing to do with resolving obscurities and difficulties!"<sup>173</sup> On the face of it, this shows a profound disregard on Ibn Sīnā's part for the underlying logical structure of mathematical argument. However, as Lokotsch aptly notes, Ibn Sīnā's purpose in summarizing the geometry in the *Shifā'* is to provide quick access to mathematical knowledge.<sup>174</sup> Similarly, his purpose in producing a geometry section in the 'Alā'ī is simply to introduce material necessary for studying the *Almagest*.<sup>175</sup> In fact, one has only to look at Ibn Sīnā's logical analysis of Euclid's first proposition in the section of the 'Alā'ī on syllogism to see that he shows ample regard for the relation between logic and mathematics.<sup>176</sup>

Before pursuing Ibn Sīnā's views further, let us briefly survey the opinions of his time on mathematics. Medieval Arab philosophers tended to take the Aristotelian view of mathematics, regarding "mathematical objects... as existing by abstraction,"<sup>177</sup> and taking the proper domain of

<sup>171</sup> Hairetdinova (41), 29.

<sup>172</sup> Heath (46), I, 77 conjectures the author to be al-Ṭūsī Bodelian MS. 280, dating from 1260-1261. Murdoch (78), 453, however, identifies it as a copy of the *Tahrīr* of al-Magh-ribī

<sup>173</sup> See Heiberg (47), 16, where Nicoll's Latin translation of the preface is reproduced in its entirety. The section we translate reads "Etenim doctor primarius (Avicenna) postulata et definitiones multas resecurit, difficilium quoque et obscurorum resolutionem *detrectavit*." Cf. Heath (46), I, 77.

<sup>174</sup> Lokotsch (67), 25, "Avicenna dagegen kam es mehr darauf an, eine Zusammenfassung der Elemente zu schaffen, die sich in Verbindung mit mündlicher Unterweisung zur schnellen Aneignung mathematischer Kenntnisse eignete".

<sup>175</sup> See section on "Geometry" above, in particular al-Jūzajānī's introduction to the 'Alā'īs resumé of Euclid.

<sup>176</sup> Zabeeh (126), 36-37.

<sup>177</sup> Aristotle, *De Caelo* III. 1, 299a 15 ff.; see Heath (45), 224-226.

mathematics to be “only the quantitative and continuous,”<sup>178</sup> not matter, and the qualities of matter like weight, hardness, heat, etc. Ibn Rushd (1126-1198), in his *Commentary on the Metaphysics of Aristotle* (Tafsīr Mā Ba’d at-Tabi’at), is a prime exponent of this view, and as Pines<sup>179</sup> points out, it is probably for similar reasons that al-Kindī, the Ikhwan al-Safa, Maimonides and Ibn Rushd regarded mathematics as providing an inferior model of the real world than that of Aristotelian physics.

Ībn Sīnā, on the other hand, appears to take a different view, especially, if one recalls his classification of the sciences, which puts mathematics ahead of physics, but just behind metaphysics.<sup>180</sup> Yet, his extended writings on the subject put him firmly in the Aristotelian camp. For example, in the Kitāb al-Najāt, he notes that the objects of mathematics are “things conceived by the estimative faculty [al-wahmiyyāt],” as opposed to things which (physically exist).<sup>181</sup> Al-Ṭūsī, in his commentary on Ibn Sīnā’s Kitāb al-Ishārāt wa’l-Tanbīhāt, goes on to assert that it is just this faculty (al-wahm) which accounts for the uniformity of opinion on what constitutes mathematics.<sup>182</sup> Moreover, somewhat later in the Najāt, Ibn Sīnā says that “the mathematical sciences are not [qualified] to investigate either the natures of physical substances or quantities inherent in these substances,”<sup>183</sup> echoing a passage in al-Kindī’s *Rasā’il al-Falsafiyya*.<sup>184</sup> Ibn Sīnā’s firm commitment to the Aristotelian view of mathematics and physics is further in evidence in his famous exchange of questions and answers with al-Bīrūnī, the *als-As ‘ilah wa’l-Ajwibah*, in which the former plays Aristotle to the latter’s Zeno.<sup>185</sup>

Even Ibn Sīnā’s view of the foundations of arithmetic shows an Aristotelian outlook, or at least a Pythagoreanism channeled through

<sup>178</sup> Aristotle, *Metaphysics* X, 31, 1061a 28 ff.

<sup>179</sup> Pines (85), 79.

<sup>180</sup> Pines (85), 79, or Ibn Sīnā : *Aqsām al-’Ulūm, Tis’ Rasā’il*, Cairo, 1908, 105 —.

<sup>181</sup> p. 158, Cairo edition (1331 AH)

<sup>182</sup> Pines (85), 83, note 16. Actually al-Ṭūsī is speaking only of geometry.

<sup>183</sup> p. 339, Cairo edition (1331 AH)

<sup>184</sup> See Pines (85), 80, or M. Abu Rida, ed. *Rasā’il al-Kindī al-Falsafiyya*, Cairo (1950), 111. See also Ibn Sīnā’s *Al-Burhān* III, 196-201. Cairo ed. 1956.

<sup>185</sup> For selections, see Heinen (49), *passim*; Nasr (81), 170-173; Nasr (82), 133-138. The complete text was edited by S. H. Nasr and M. Mohaghegh : *Al-Bīrūnī and Ibn Sīnā, als-As ‘ilah wa’l-Ajwibah*, Tehran, 1973.

Aristotle, and modified by the Ikwan al-Şafā and Nicomachus. His position in analysis, that is, his view of infinity and infinitesimals, also shows the influence of Aristotle, albeit with some modifications. We close this section with a detailed examination of this position.

As does Aristotle,<sup>186</sup> Ibn Sīnā argues against the actual infinite. The argument appears in the Najāt, as well as in the section on metaphysics of the ‘Alā’ī, in a discussion of the question of efficient causes (‘illat-hā-i fā’il):<sup>187</sup>



Figure 13

“Let there be an infinite line AB. Let us indicate on it the point J and consider the line JD a finite measure or a finite quantity. If the line DB is finite, and we augment it by JD, then JB will also be finite. If the line DB is infinite, let us allow DB to correspond to JB until they are at the same place. If DB should correspond to JB, then a lesser and a greater would correspond, which is impossible, DB being the lesser and JB being the greater. If DB stands still while JB continues, then the end-point B will be limited, and JB will be greater than DB by a measure equal to JD which is finite. Thus, JB is also finite. It becomes evident, then, that such a number and such a measure are not infinite.”

Al-Shahrastani (d. 1153) reproduced this argument of Ibn Sīnā in a slightly clearer form in the Kitāb al-Milal wal-Niḥal,<sup>188</sup> and through al-Tabrīzī (2nd half, 13th c.),<sup>189</sup> it finds its way into Hasdai Crescas’ (1340-1410) *Critique of Aristotle*.<sup>190</sup> It also appears in the 1593 Rome dual edition of the Canon and al-Najāt.<sup>191</sup>

<sup>186</sup> Aristotle, *Physics*, iii, 5, 204a, 20-29; *Metaphysics* V, eh 11, 1073a 25. See also Maziarz and Greenwood (73), 173-183; Heath (45), 102-113.

<sup>187</sup> Morewedge (77), 44. al-Najāt (Cairo ed., 1938), 124-125. Also al-Ishārāt wa l-Tanbīhāt (J. Forget ed., Leiden, 1892), 94-95. See also Ibn Rushd : *Tahāfut al-Tahāfut* (Bouyges ed., Beirut, 1930), 27-28.

<sup>188</sup> (Book of Religious and Philosophical Sects), ed. W. Cureton, London, 1846, p. 403.

<sup>189</sup> See Wolfson’s edition of Crescas’ *Critique of Aristotle* (124).

<sup>190</sup> *Ibid.*, see his Index of Citations

<sup>191</sup> *Ibid.*

However, as Marmura observes, Ibn Sīnā was obliged to modify his stance in his treatise *Risāla Aḏḥawīyya fī Amr al-Ma’ād*.<sup>192</sup> In effect, Ibn Sīnā admits the existence of infinitely many *immaterial* bodies, in this case souls, arguing that since they are immaterial, they are also unordered. The implication is that his previous argument on infinity then does not apply. Evidently, Al-Ghazali (1058-1111), in *Tahāfut al-Falāsifa*,<sup>193</sup> rejects this argument, asserting that the immaterial bodies (souls) may be ordered just as the sequence of days.

Interestingly enough, none of these reworkings of Aristotle’s argument ever arrives at the notion set forth in Adam of Balsham’s (Parvipontanus) *Ars Disserendi*, that a proper subset of an (infinite) set may be equinumerous with it,<sup>194</sup> much less Galileo’s discovery that the natural numbers and their squares are in one-to-one correspondence.<sup>195</sup>

Specific instances of Ibn Sīnā’s position on infinitesimals and limits are somewhat more difficult to give. As noted in section two, the treatise “al-zawī-ya” on the “hornlike” angle may contain some relevant passages; however, we have been unable to acquire a copy of the manuscript.<sup>196</sup> This leaves als-As ‘ilah ‘wa’l-Ajwibah; a few of its exchanges are sufficient to give the general direction of Ibn Sīnā’s thoughts on the infinitesimal. Al-Bīrūnī asks : “If a square is divided into four equal squares, how are the surfaces of the squares at opposite coners contiguous with each other?”<sup>197</sup> Ibn Sīnā’s answer is built on two Aristotelian precepts : “a point cannot be contiguous with a point,”<sup>198</sup> and “a line is not made up of points.”<sup>199</sup> In a

<sup>192</sup> Marmura (71), especially 233-234. The *Risāla Aḏḥawīyya fī Amr al-Ma’ād* was edited by Dunya, Cairo, 1949.

<sup>193</sup> For specific citations to *Tahāfut al-Falāsifa* (M. Bouyges, ed., Beirut, 1927), see Marmura (71), 236. We restrict our discussion to the mathematical aspects of al-Ghazālī’s argument; the interested reader will find that the principle issue, is, however, a religious one.

<sup>194</sup> Thomas (113), 133-134, and I. L. Minio-Paluello’s *Adam Balsamiensis Parvipontani Ars Disserendi*, Rome, 1956. See also Brown (15).

<sup>195</sup> Struik (108), 198-207; especially 205, and Drake (30), 28-47.

<sup>196</sup> Patna (Bankipore) Oriental Public Library; Patna II, 456, 2361, 6. (Brockelmann I, p. 596 no. 79r).

<sup>197</sup> Heinen (49), 53-54.

<sup>198</sup> *De generatione et corruptione* I. 2. 317<sup>a</sup> 11: οὐ γάρ ἐστιν ἐχόμενον σημεῖον σημείου ἢ στιγμὴ στιγμῆς.

<sup>199</sup> *Physika* IV. 8. 215<sup>b</sup> 19 : οὐ σύγχειται ἐξ στιγμῶν.

second passage (answer to al-Bīrūnī's question4),<sup>200</sup> Ibn Sīnā bases his reply on Aristotle's position that infinite subdivision exists only potentially,<sup>201</sup> while al-Bīrūnī counters that such an answer implies that the side of a square and its diagonal are of equal length because they are, potentially, equinumerous.<sup>202</sup> Of course, to do justice to both Ibn Sīnā and al-Bīrūnī we must note that these arguments are more about the classical Atomistic theory than about mathematics. Similar discussions appear in the works of al-Aš'arī (873/4-935/6) and al-Baqīlanī (d. 1013), who seem to espouse the view that the side of a square and its diagonal are commensurable.<sup>203</sup>

The other notion of infinitesimal, based on the principle of Eudoxus (axiom of Archimedes),<sup>204</sup> and used so effectively by Archimedes in his treatise : Quadrature of the Parabola, and On the Sphere and Cylinder, and emulated by Thābit Ibn Qurra, al-Kūhī and Ibn al-Hāyṯham<sup>205</sup> appears in Ibn Sīnā's writings only peripherally. For example, he includes Euclid's version of it in the geometric section of the *Shifā'* (Elements V, dfn. 5).

## VII. Summary

As we remarked in part I, Sarton's off hand dismissal of Ibn Sīnā as a mathematician gave partial impetus to this study. Recall those words "Ibn Sīnā's interest in mathematics was philosophical rather than technical,... [as to be expected] in a late Neoplatonist." We are inclined to feel that this opinion is rather unfounded. Certainly Ibn Sīnā's work on mathematics goes much beyond the traditional philosophical manuals, such as Theon of Smyrna's *Mathematics Useful for Understanding Plato*, occasionally improves on its sources, as is in evidence by his treatment of topics in common with Nicomachus, and often has a fixed, practical purpose: the arithmetic of the 'Alā'ī serves to introduce musical theory, and its geometry prepares the reader for a study of the *Almagest*.

<sup>200</sup> Nasr (82), 134.

<sup>201</sup> *Ibid.*, 135; Physika III. 6. 206<sup>a</sup> 9-<sup>b</sup>27; see Heath (45), 105-107.

<sup>202</sup> *Ibid.*, 135-136, following S. H. Nasr's translation.

<sup>203</sup> Juschkewitsch (54), 204-205.

<sup>204</sup> Physika III. 7. 207<sup>a</sup> 33-<sup>b</sup> 34; see Heath (45), 110-112.

<sup>205</sup> For an excellent survey of this, see Juschkewitsch (54), 288-295.

Of course, others have criticized Ibn Sīnā's style of presenting his material, as well as his philosophical bias. Al-Bīrūnī, in a treatise on chords,<sup>206</sup> notes that Ibn Sīnā regarded any collection of assorted proofs of the same theorem “fūdhūl” [redundant],<sup>207</sup> while the writer of the preface to a 13th century edition of the *Elements* (Nicoll, 280), accuses him of leaving out postulates and axioms and neglecting to address the difficulties of the subject.<sup>208</sup> We have already discussed the latter accusation. As for al-Bīrūnī's comment, suffice it to note that lemmas, porisms and alternate proofs were regarded as “fūdhūl” in most Arabic mathematical writings, notably in the versions of the *Elements*.<sup>209</sup>

On the other hand, one cannot altogether dismiss Ibn Funduq's observation that a person like Ibn Sīnā wasn't much of a mathematician because he'd tasted the sweetness of metaphysics and fancied mathematics from time to time, only to abandon it.<sup>210</sup> Ibn Sīnā's mathematics certainly has none of the originality of Thābit's, nor even the care for detail of Ishāq. The geometry probably derives from the translation of al-Hajjāj, the arithmetic from Nicomachus and certain Indian sources, the trigonometry from al-Fārābī, the philosophy of mathematics and notions on its foundations from Aristotle, possibly through al-Kindī and al-Fārābī. Still, the arrangements, and many of the details are Sīnā's own. Moreover, as al-Jūzajani points out, mathematics, a subject in which Ibn Sīnā was mostly self-taught, and only one of many subjects that occupied him, did so for his entire life.

<sup>206</sup> Igtikhrāj al-Awtār fī al-Dāirah bi-khawāṣṣ al-khaṭṭ al-Munhanī fihā. See Saidan (93), 683.

<sup>207</sup> Igtikhrāj al-Awtār fī al-Dāirah bi-khawāṣṣ al-khaṭṭ al-Munhanī fihā. See Saidan (93), 683.

<sup>208</sup> See note 173 above.

<sup>209</sup> Heath (46), I, 81.

<sup>210</sup> See note 18 above.

**Appendix I (A)**  
**K. al-Shifā'-Geometry**  
**(Preliminary identification of contents)**

**Divisions of text into books**

1. Definitions of triangle and parallelogram (12)<sup>1</sup>
2. The straight line, segments and applications thereof (5)
3. Circles (10)
4. Applications of triangles and circles (5)
5. Ratios (6)
6. Similar figures (7)
7. Common factors, inequalities and their connetions (10)
8. Proportions (5)
9. Proportions and related notions, factors, etc. (6)
10. [Same as 7] (17)
11. Solid geometry (6)
12. Polyhedrons (3)
13. Mean proportionals and regular polygons (4)
14. Mean proportionals and regular polyhedrons (3)
15. Inscribing regular polyhedrons inside each other (1)

The text of the geometric section then follows, in 433 pages in the Sabra edition. There is no introduction, preface, or statement of purpose, in contrast to al-handasa section of the 'Alā'ī.

Book I of al-handasa begins with the Euclidean definitions of point, line; some common notions ('ilm gami') but no postulates other than postulate 5 are initially listed.

<sup>1</sup> Numbers in parenthesis in 'Divisions ...' 1-15 give the approximate percentage of the text of al-Handasa devoted to each division. (Sabra ed.)

<b>Shifā'<sup>2</sup></b>	<b>Correspondence Elements</b>
20, 1	I, 1
21, 2	I, 2
22, 3 } 23, 4 } 24, 5 }	I, postulate 3
25, 6	I, 3
26, 7	I, 4
27, 8	I, 5
28, 9	I, 6
29, 10 <sup>3</sup>	I, 7
30, 11 <sup>4</sup>	—
31, 12	I, 8
32, 15 <sup>5</sup>	—
33, 14	I, 9
34, 15	I, 10
34-35, 16	I, 11
35, 17	I, 12
36, 18	I, 13
37, 19 <sup>6</sup>	I, 14
38, 20	I, 15
38, 21 <sup>7</sup>	—
39, 22 <sup>8</sup>	I, 16
40, 23	I, 17
41, 24	I, 18
41, 25	I, 19

<sup>2</sup> Numbers under Shifā' refer to page and proposition numbers in the Sabra edition; those under *Elements* are the corresponding book and proposition number in Heath's edition. However, see note 60, below.

<sup>3</sup> However, the illustration is for the second case of the proposition, where D lies in the  $\triangle ABC$ , i. e., the case treated by Proclus.

<sup>4</sup> The text is jumbled, and makes little sense.

<sup>5</sup> This is a preliminary to I, 9. The figure is Proclus' second case objection to I, 9.

<sup>6</sup> Ibn Sīnā's figure is at slight variance with the usual Euclidean figure, i. e., Heath's. Similar comments later in these notes are made with the same frame of reference.

<sup>7</sup> This is a 'converse' to I, 15, namely, "if the vertical angles are equal, the lines are straight, etc".

<sup>8</sup> Ibn Sīnā's figure shows a right angled triangle, but his statement of the proposition is not restricted.

41-42, 26 <sup>9</sup>	I, 20
42, 27	I, 21
43, 28 <sup>10</sup>	I, 22
44, 29	I, 23
45, 30	I, 24 <sup>11</sup> -I, 25
46, 31	
47-48, 32	
48, 33	I, 26
49, 34	I, 27
50, 35	I, 28
51, 36	I, 29
51, 37	I, 30
52, 38 <sup>12</sup>	I, 31
52, 39	I, 32
<b>Shifā'</b>	<b>Elements</b>
53, 40	I, 33
53-55, 41 <sup>13</sup>	I, 34-1, 35
56, 43	I, 36
56, 44	I, 37
57, 45	I, 38
58, 46	I, 39
58-59, 47	I, 41
59, 48	I, 42
60, 49	I, 43
61, 50	I, 44
62, 51	I, 46 <sup>14</sup>
63-64, 52 <sup>15</sup>	I, 47
65, 53	I, 48

<sup>9</sup> The figure numbered 25 on p. 42 is really for 26.

<sup>10</sup> The Euclidean caveat-it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one -is included in Ibn Sīnā's version.

<sup>11</sup> The figure for I, 24 resembles Proclus' first of two cases.

<sup>12</sup> The figure seems to suggest that Ibn Sīnā is taking a perpendicular transversal, but the text shows no such restriction.

<sup>13</sup> There is no 42.

<sup>14</sup> I, 45 is also missing from the al-Hajjāj Ibn Yūsūf translation. See Heath (46), I, 76, and Besthorn (11), Partis I, fasc. I. -II.

<sup>15</sup> The figure is nearly Euclid's. There is no mention of al-Nayrīzī, or of Thābit's generalization of the Pythagorean theorem. (See Sayili (95).

<b>Shifā' (Book 2)<sup>16</sup></b>	<b>Elements</b>
69, 1	II, 1
70, 2	II, 2
71, 3	II, 3
71-72, 4	II, 4
73, 5	II, 5
74, 6	II, 6
75, 7	II, 7
75-76, 8	II, 8
77, 9 } 78, 9 } 79, 9 }	II, 9
79, (10) } 80, 12 }	II, 10
81, 13 } 81, 14 }	II, 11
82, (12) <sup>17</sup> } 83, (15) }	II, 12
83, (13) <sup>18</sup> } 84, 16 } 85, 17 }	II, 13
<b>Shifā' (Book 2)</b>	<b>Elements</b>
85, 14 <sup>19</sup> } 86, 18 }	II, 14
<b>Shifā' (Book 3)<sup>20</sup></b>	<b>Elements</b>
90, 1	III, 1
91, 2	III, 2
92, 3	III, 3
93, 4	III, 4
93-94, 5	III, 5
94, 6	III, 6
95, 7	III, 7

<sup>16</sup> Again, Ibn Sīnā begins with definitions.

<sup>17</sup> 82, (12) is an introduction to 83, (15). It treats the (impossible) case of the perpendicular lying "outside".

<sup>18</sup> 83, (13) plays the same preliminary role for 84, 16.

<sup>19</sup> Ibn Sīnā restricts himself to triangles (rectilinear figures).

<sup>20</sup> The book is introduced by the Euclidean definitions from Elements, III.

96, 8	III, 8
98, 9	III, 9
98, 10 <sup>21</sup>	III, 10
100, 11	III, 11-12
101, 12 <sup>22</sup>	III, 13
102, 13	III, 14
103, 14	III, 15
104, 15 <sup>23</sup>	III, 16 <sup>—</sup>
105, 16	III, 17
106, 17 <sup>24</sup>	III, 18-III, 19
107-108, 18 <sup>25</sup>	III, 20
109, 19	III, 21
109, 20	III, 22
110, 21 <sup>26</sup>	III, 23-III, 24
111, 22 <sup>27</sup>	—
111, 23	III, 25
113, 25 <sup>28</sup>	III, 27
114, 26 <sup>29</sup>	—
114-115, 27	III, 28
115, 28	III, 30
116, 29	III, 31
117-119, 30	III, 32
119-120, 31 <sup>30</sup>	III, 33
121, 32	III, 34

<sup>21</sup> Ibn Sīnā's figure suggests a symmetry not in Euclid; but again the treatment is general.

<sup>22</sup> III, 12 is not in the al-Hajjāj translation either; see Heath (46), I, 80, or Besthorn (11), Partis II.

<sup>23</sup> It is of interest to compare this proposition with Ibn Sīnā's manuscript "al-zāwiya". See note 196 of the text of this essay.

<sup>24</sup> The figure and statements for III, 19 appear after 106, 17 without any separate numbering in Sabra's edition.

<sup>25</sup> The three figures in Ibn Sīnā represent decompositions of the usual Euclidean figure.

<sup>26</sup> 110, 21 is followed, without separate numbering, by a figure and statement of III, 24.

<sup>27</sup> Part of III, 24.

<sup>28</sup> There is no 24 in Sabra's edition.

<sup>29</sup> Part of III, 27.

<sup>30</sup> Ibn Sīnā considers the three cases : right, obtuse and acute angle.

<b>Shifā' (Book 3)</b>	<b>Elements</b>
122-126, 33-34 <sup>31</sup>	III, 35
127, 35 <sup>32</sup>	III, 36
129, 36	III, 37

<b>Shifā' (Book 4)<sup>33</sup></b>	<b>Elements</b>
133, 1	IV, 1
134, 2	IV, 2
134-135, 3	IV, 3
136, 4	IV, 4
137, 5 <sup>34</sup>	—
138, 6	—
139, 7	IV, 5
140, 8	IV, 6
140-141, 9	IV, 7
141, 10	IV, 8
142, 11	IV, 9
142-143, 12 <sup>35</sup>	IV, 10
143-144, 13	IV, 11
144-145, 14	IV, 12
146, 15	IV, 13
147, 16	IV, 14
148, 17	IV, 15
148-150, 18	IV, 16

<b>Shifā' (Book 5)</b>	<b>Elements</b>
155, 1	V, 1
156, 2	V, 2
157, 3	V, 3
158, 4	V, 4
159, 5	V, 5
159-160, 6	V, 6
160-161, 7	V, 7
161-162, 8	V, 8

<sup>31</sup> Ibn Sīnā gives four figures; compare with Campanus' four cases and Euclid's two.

<sup>32</sup> There is no suggestion that Ibn Sīnā sees the tangent case as the limit of the secant case in his treatment of III, 36.

<sup>33</sup> No definitions introduce this book.

<sup>34</sup> 137, 5 and 138, 6 serve to introduce 139, 7.

<sup>35</sup> Ibn Sīnā's figure departs somewhat from the usual Euclidean one.

163, 9	V, 9
163, 10 <sup>36</sup>	V, 10
<b>Shifā' (Book 5)</b>	<b>Elements</b>
164, 11	V, 11
164-165, 12	V, 13
165-166, 13 <sup>37</sup>	V, 12
166, 14	V, 14
166-167, 15	V, 15
167, 16	V, 16
168, 17 <sup>38</sup>	—
169-170, 18	V, 17
171, 19	V, 18
172, 20	V, 19
173, 21 <sup>39</sup>	—
174, 22	V, 20
175, 23	V, 24
175-176, 24 <sup>40</sup>	—
176, 25	V, 25
<b>Shifā' (Book 6)<sup>41</sup></b>	<b>Elements</b>
179, 1 <sup>42</sup>	VI, 1
181, 2	VI, 1
181-182, 3	VI, 2
182-183, 4	VI, 3
184, 5	VI, 4
185, 6	VI, 5
186, 7	VI, 6
187, 8	VI, 8
188, 9 <sup>43</sup>	—
188-189, 10	—
189, 11	VI, 13

<sup>36</sup> Ibn Sīnā appears to have two figures (in Sabra's edition) for 163, 10. The first of these, however, is for 163, 9.

<sup>37</sup> The single figure serves both 164-5, 12 and 165-6,

<sup>38</sup> This serves to introduce 169-170, 18.

<sup>39</sup> This introduces 174, 22.

<sup>40</sup> Introduction to 176, 25.

<sup>41</sup> The book begins with definitions of similar and congruent figures.

<sup>42</sup> This is VI, 1 modified to "triangles and squares....",

<sup>43</sup> 188, 9 and 188-189, 10 are cases of 187, 8 (= VI, 8).

190, 12	VI, 11
190, un-numbered	VI, 14
191, 13	VI, 15
191-192, 14	VI, 10
192-193, 15	VI, 17
193, 16	VI, 19
194, 17 <sup>44</sup>	—
<b>Shifā' (Book 6)</b>	<b>Elements</b>
195, 18	VI, 20
195-196, 19	VI, 18
196-197, 20	VI, 21
197, 21	VI, 22
198, 22	VI, 24
198-199, 23	VI, 26
199-200, 24 <sup>45</sup>	VI, 25
200-201, 25	VI, 23
201-202, 26 } 202-203, 27 }	VI, 28
204, 28	VI, 30
204-205, 29	VI, 32
205-206, 30	VI, 31
206-207, 31	VI, 33
<b>Shifā' (Book 7)<sup>46</sup></b>	<b>Elements</b>
213, 1	VII, 1
214-215, 2	VII, 2
215, 3	VII, 3
216, 4	VII, 4
217, 5	VII, 5
217-218, 6	VII, 6
218, 7	VII, 7
219, 8	VII, 8
220, 9	VII, 9
221, 10	VII, 10
222, 11	VII, 11
222-223, 12	VII, 12

<sup>44</sup> Introduces 195, 18.

<sup>45</sup> This appears to be part of VI, 28.

<sup>46</sup> This book is prefaced by Euclidean definitions

223, 13	VII, 13
223-224, 14	VII, 14
224-225, 15 <sup>47</sup>	VII, 15
225, 16 <sup>48</sup>	VII, 16
226, 17 <sup>49</sup>	VII, 17
226, 18 <sup>50</sup>	VII, 18
227, 19	VII, 19
227-228, 20	VII, 20
<b>Shifā' (Book 7)</b>	<b>Elements</b>
228-229, 21	VII, 22
229, 22	VII, 21
229, 23	VII, 23
229-230, 24	VII, 24
230-231, 25	VII, 25
231, 26	VII, 26
231-232, 27	VII, 27
232, 28	VII, 28
233, 29	VII, 31
233, 30 <sup>51</sup>	VII, 32
233, 31	VII, 29
234, 32	VII, 30
235, 33	VII, 33
236, 34	VII, 34
237, 35 <sup>52</sup>	—
238, 36	VII, 35
238, 27	VII, 36
239, 38 <sup>53</sup>	—
239-240, 29	VII, 37
240, 40	VII, 38
240-241, 41	VII, 39

<sup>47</sup> In each case Ibn Sīnā's figure differs from the Euclidean one.

<sup>48</sup> In each case Ibn Sīnā's figure differs from the Euclidean one.

<sup>49</sup> In each case Ibn Sīnā's figure differs from the Euclidean one.

<sup>50</sup> No figure.

<sup>51</sup> The identification is only probable since the text is unclear.

<sup>52</sup> Part of 236, 34 (VII, 34).

<sup>53</sup> Part of 238, 34 (VII, 36).

<b>Shifā' (Book 8)</b>	<b>Elements</b>
245, 1	VIII, 1
246, 2	VIII, 2
247, 3	VIII, 3
248, 4	VIII, 4
249, 5	VIII, 5
250, 6	VIII, 6
251, 7	VIII, 7
252-253, 8	VIII, 8
253-254, 9 <sup>54</sup>	VIII, 9
254-255, 10	VIII, 10
256, 11	VIII, 12
257, 12	VIII, 13

<b>Shifā' (Book 8)</b>	<b>Elements</b>
258, 13	VIII, 14
258-259, 14	VIII, 15
259, 15	VIII, 16
260, 16	VIII, 18
260-261, 17	
261-262, 18	VIII, 19
263, 19	VIII, 21
264, 20	VIII, 22
264-265, 21	VIII, 23
265, 22	VIII, 24
265, 23	VIII, 25
265-266, 24	VIII, 26
266-267, 25	VIII, 27

<b>Shifā' (Book 9)<sup>55</sup></b>	<b>Elements</b>
271, 1	IX, 1
271-272, 2	IX, 2
272, 3	IX, 3
273, 4	IX, 4
273, 5	IX, 5
274, 6	IX, 6
274, 7	IX, 7
275, 8	IX, 8

<sup>54</sup> Ibn Sīnā's figure differs from the Euclidean one.

<sup>55</sup> As in the Elements, no definitions introduce this book.

276, 9	IX, 9
277, 10	IX, 10
277-278, 11	IX, 11
278, 12	IX, 12
279, 13	IX, 13
280-281, 14	IX, 14
281-282, 15	IX, 15
283, 16	IX, 16
283-284, 17	IX, 17
284-285, 18	IX, 18
285, 19	IX, 19
286, 20	IX, 20
<b>Shifā' (Book 9)</b>	<b>Elements</b>
287, 21	IX, 21
287, 22	IX, 22
288, 23	IX, 23
288, 24	IX, 24
289, 25	IX, 26
289, 26	IX, 27
289-290, 27	IX, 25
290, 28	IX, 28
290-291, 29	IX, 29
291, 30	IX, 30
291-292, 31	IX, 31
292, 32	IX, 32
293, 33	IX, 33
293, 34	IX, 34
293-294, 35	IX, 35
294, 36	IX, 36
<b>Shifā' (Book 10)<sup>56</sup></b>	<b>Elements</b>
299-300, 1 <sup>57</sup>	X, 1
301, 2	X, 2
302, 3	X, 3
303, 4	X, 4
303-304, 5	X, 5
304-305, 6	X, 6

<sup>56</sup> Ibn Sīnā gives the four Euclidean definitions.

<sup>57</sup> There is a difference between Ibn Sīnā's figure and the Euclidean one.

305-306, 7 }	X, 9
306, 8 }	
306-307, 9	X, 12
307, 10	X, 15
307, 11	X, 11
308, 12	X, 17
309, 13	X, 14
310, 14 }	
311, 15 }	X, 18
312, 16	X, 20
312, 17	X, 21

**Shifā' (Book 10)**

313, 18 }	
313-314, 19 }	
315, 20	X, 13
315-316, 21	X, 22
316-317, 22	X, 23
318, 23	X, 24
319, 24 <sup>58</sup>	X, 25
320-321, 25	X, 26
322, 26	X, 28
323, 27	X, 33
323-324, 28	X, 34
324, 29	X, 35
325, 30 <sup>59</sup>	—
325-326, 31	X, 36
326, 32	X, 37
326-327, 33	X, 38
327-328, 34	X, 39
328, 35	X, 40
328-329, 36	X, 41
329, 37	X, 42
329-330, 38	X, 43
330-331, 39	X, 44
331, 40	X, 45
331, 41	X, 46

**Elements**

Lemma between X, 13, and X, 14

<sup>58</sup> Apparently Ibn Sīnā replaces rectangles by squares in his enunciation of 319, 24

<sup>59</sup> Introduces 325-326, 31.

331-332, 42	X, 47
333, 43	X, 48
333, 44	X, 49
333, 45	X, 50
333, 46	X, 51
334-335, 47	X, 52
335, 48	X, 53
335-336, 49	X, 54
337-338, 50	X, 55
338, 51	X, 56
338-339, 52	X, 57
339, 53	X, 58

**Shifa' (Book 10)**

339, 54	<b>Elements</b> X, 59
339-340, 55 } 340-342, 56 }	X, 60
343, 57	X, 61
343, 58	X, 62
344, 59	X, 63
344, 60	X, 64
344, 61	X, 65
345, 62	X, 66
345-346, 63	X, 67
346, 64	X, 68
347, 65	X, 69
347, 66	X, 70
347-348, 67	X, 71
349, 68	X, 72
349-350, 69	X, 73
350, 70	X, 74
350-351, 71	X, 76
352, 72	X, 75
353, 73	X, 77
353, 74	X, 78
353, 75	X, 79
353, 76	X, 80
354-355, 77	X, 81

355, 78	}	X, 82
355, 79		
355-356, 80		
356, 81		X, 85
357, 82		X, 86
358-359, 83		X, 87
358, 84		X, 88
358, 85		X, 89
358, 86		X, 90
359-360, 87		X, 91
361, 88		X, 92
361, 89		X, 93
361-362, 90		X, 94

**Shifā' (Book 10)**

**Elements**

362, 91	X, 95
362, 92	X, 96
362-363, 93	X, 97
364, 94	X, 98
364, 95	X, 99
364-365, 96	X, 100
365, 97	X, 101
365, 98	X, 102
365-366, 99	X, 103
366, 100	X, 104
366-367, 101	X, 105
367, 102	X, 106
367, 103	X, 107
367-368, 104	X, 108
368, 105	X, 109
368-369, 106	X, 110
369-370, 107	X, 111
370-371, 108	X, 115

<b>Shifā' (Book 11)<sup>60</sup></b>	<b>Elements</b>
376, 1 (f. 330)	XI. 1
377, 2 (f. 331)	XI. 2
377, 3 (f. 332)	XI. 3
378, 4 (f. 333)	XI. 4
379, 5 (f. 334)	XI. 5
380, 6 (f. 335)	XI. 6
380, 7 (f. 336)	XI. 7
381, 8 (f. 337) <sup>61</sup>	XI. 8
382, 9 (f. 338)	XI. 9
382, 10 (f. 339)	XI. 10
382, 11 (f. 340)	XI. 11
383, 12 (f. 341)	XI. 12
383, 13 (f. 342)	XI. 13
384, 14 (f. 343)	XI. 14
384, 15 (f. 344)	XI. 15

<b>Shifā' (Book 11)</b>	<b>Elements</b>
385, 16 (f. 345)	XI. 16
385, 17 (f. 346)	XI. 17
386, 18 (f. 347)	XI. 18
386, 19 (f. 348)	XI. 19
387, 20 (f. 349)	XI. 20
388, 21 (f. 350)	XI. 21
388, 22 (f. 351)	XI. 22
389, 23 (f. 352)	XI. 24
389, 24 (f. 353)	XI. 25
390, 25 (f. 354)	XI. 26
390, 26 (f. 355)	XI. 27
391, 27 (f. 356)	XI. 28
391, 28 (f. 357)	XI. 29
392, 29 (f. 358)	XI. 30
392, 30 (f. 359) <sup>62</sup>	XI. 31

<sup>60</sup> Here we must change our manner of identifying the propositions in the *Shifā'*, since neither Ibn Sīnā, nor his editōr supply numbers. Thus 376, 1 (f. 330) is a proposition on p. 376 with figure 330. Our label 1 is an interpolation which we change consecutively, e. g., the next proposition is 377, 2 (f. 331), etc. The book begins with Euclid's definitions from solid geometry, as Elements XI.

<sup>61</sup> This figure as well as figure 337 show a total lack of perspective, an element worth considering in analysis of Arabic mss. of Euclid.

<sup>62</sup> Figure differs from Euclid.

393, 31 (f. 360)	XI. 32
393, 32 (f. 361) <sup>63</sup>	—
394, 33 (f. 362) <sup>64</sup>	—
395, 34 (f. 363) <sup>65</sup>	XI. 33
395, 35 (f. 364)	XI. 35
396, 36 (f. 365)	XI. 36
397, 37 (f. 366) <sup>66</sup>	XI. 37
398, 38 (f. 367) <sup>67</sup>	XI. 38
398, 39 (f. 368)	XI. 39

<b>Shifā' (Book 12)<sup>68</sup></b>	<b>Elements</b>
401, 1 (f. 369)	XII, 1
402, 2 (f. 370) <sup>69</sup>	—
402, 3 (f. 371)	XII, 2
403, 4 (f. 372)	XII, 3
404, 5 (f. 373)	XII, 4
405, 6 (f. 374) <sup>70</sup>	XII, 5
405, 7 (f. 375)	XII, 6
406, 8 (f. 376) <sup>71</sup>	XII, 7
406, 9 (f. 377) <sup>72</sup>	XII, 9

<b>Shifā' (Book 12)</b>	<b>Elements</b>
407, 10 (f. 378) <sup>73</sup>	XII, 10
408, 11 (f. 379) <sup>74</sup>	XII, 12
409, 12 (f. 380)	XII, 11
409, 13 (f. 381)	XII, 15
410, 14 (f. 382)	XII, 16

<sup>63</sup> 393, 32 is essentially the planar version of XI, 32; see VI, 1, etc. It and 394, 33 can be viewed as an introduction to 395, 34 (XI, 33). Literally 393, 32 is : "parallelograms of the same height have proportional bases".

<sup>64</sup> See n. 63. Literally 394, 33 is : "equal parallelograms of equal base have equal height".

<sup>65</sup> Figure is different from Euclid.

<sup>66</sup> The figure is that from Heiberg's ms b; see Heath (46), III, 359.

<sup>67</sup> Figure differs from Euclid.

<sup>68</sup> Following *Elements* XII, there are no introductory definitions.

<sup>69</sup> 402, 2 serves to introduce 402, 3

<sup>70</sup> Figure differs slightly from Euclid. Cone is confused with pyramid in the text of 406, 9.

<sup>71</sup> Figure differs slightly from Euclid. Cone is confused with pyramid in the text of 406, 9.

<sup>72</sup> Figure differs slightly from Euclid. Cone is confused with pyramid in the text of 406, 9.

<sup>73</sup> Figure differs slightly from Euclid. Cone is confused with pyramid in the text of 406, 9.

<sup>74</sup> Figure differs slightly from Euclid. Cone is confused with pyramid in the text of 406, 9.

411, 15 (f. 383)	XII, 17
412, 16 (f. 384)	XII, 18
<b>Shifā' (Book 13)<sup>75</sup></b>	<b>Elements</b>
415, 1 (f. 385)	XIII, 1, 2
416, 2 (f. 386)	XIII, 3
416, 3 (f. 388) <sup>76</sup>	—
417, 4 (f. 389)	XIII, 6
417, 5 (f. 390) <sup>77</sup>	—
418, 6 (f. 391)	XIII, 12
418, 7 (392)	XIII, 9
419, 8 (f. 393)	XIII, 10
420, 9 (f. 394)	XIII, 8
421, 10 (f. 395)	XIII, 11
422, 11 (f. 396)	XIII, 13
422, 12 (f. 397)	XIII, 15
423, 13 (f. 398)	XIII, 14
424, 14 (f. 399)	XIII, 16
425, 15 (f. 400)	XIII, 17
426, 16 (f. 401) }	XIII, 18
427, 17 (f. 402) }	
<b>Shifā' (Book 14)<sup>78</sup></b>	<b>Hypsicles</b>
433, 1 (f. 403) <sup>79</sup> }	XIV, 1 <sup>80</sup>
434, 2 (f. 404) }	
435, 3 (f. 405)	Lemma (following prop. 2) <sup>81</sup>
435, 4 (f. 406) <sup>82</sup> }	XIV, 6
435, 5 (f. 407) }	

<sup>75</sup> Again, following *Elements* XIII, there are no introductory definitions.

<sup>76</sup> There is no figure 387. Figure 386 serves to introduce 417, 4.

<sup>77</sup> This introduces 418, 6.

<sup>78</sup> Book 13 concludes the *Elements* proper; however, Ibn Sīnā goes on to include the so-called *Elements* XIV of Hypsicles, followed by *Elements* XV (possibly by Isidorus of Miletus and Eutocius). See Heath (46), III, 519-520.

<sup>79</sup> 433, 1 introduces 434, 2.

<sup>80</sup> Designation follows Heath (46), III, 512 ff.

<sup>81</sup> "If an equilateral and equiangular pentagon be inscribed in a circle, the sum of the squares on the straight line subtending two sides and on the side of the pentagon is five times the square on the radius". (Heath (46), III, 519).

<sup>82</sup> This introduces 435, 5

436, 6 (f. 408)	XIV, 6 <sup>83</sup>
436, 7 (f. 409)	XIV, 6 <sup>84</sup>
Shifā' (Book 14)	Hypsicles
438, 8 (f. 410)	XIV, 7
438, 9 (f. 411)	XIV, 8
439, 10 (f. 412)	XIV, 8 <sup>85</sup>
440, 11 (f. 413)	XIV, 8 <sup>86</sup>
<b>Shifā' (Book 15)</b>	<b>[Isidorus and Eutocius?]<sup>87</sup></b>
445, 1 (f. 414) <sup>88</sup>	XIV, 3 <sup>89</sup>
445, 2 (f. 415)	XIV, 2 <sup>90</sup>
446, 3 (f. 416)	XIV, 1 <sup>91</sup>
447, 4 (f. 417)	XIV, 4 <sup>92</sup>
448, 5 (f. 418)	XIV, 5 <sup>93</sup>

## Appendix I (B)

### Ishāq-Thābit Elements

- I. 48 propositions
- II. 14 propositions
- III. 36 propositions III, 11, 12 as one prop.
- IV. 16 propositions (But text has 17 prop.; stili count accurate since IV 15, 16 are proofs of some result)
- V. 25 propositions order VI 13; V 12
- VI. 33 propositions

<sup>83</sup> Actually this is Hypsicles second proof of XIV, 6 including the "preliminary," see Heath (46), III, 516-517.

<sup>84</sup> This completes the second proof.

<sup>85</sup> This is another version of XIV, 8.

<sup>86</sup> Actually, this is the second part of the statement of XIV, 8 involving the "content" ratio.

<sup>87</sup> See note 78 above.

<sup>88</sup> Since Heath's description of *Elements XV* (Heath (46), III, 519-520) is much abbreviated, our identifying reference will be to the Heiberg and Menge edition (48), V, 40-67.

<sup>89</sup> Heiberg and Menge (48), V, 43; to inscribe an octahedron in a cube.

<sup>90</sup> Heiberg and Menge (48), V, 41; to inscribe an octahedron in a pyramid.

<sup>91</sup> *Ibid.*, 41; to inscribe a pyramid in a cube.

<sup>92</sup> *Ibid.*, 45; to inscribe a cube in an octahedron.

<sup>93</sup> *Ibid.*, 47; to inscribe a dodecahedron in an icosahedron.

order : 1-8, 13, 11, 12, 9, 10, 14-17, 19, 20, 18, 21, 22,  
24, 26, 23, 25, 27-30, 32, 31, 33.

- VII. 39 propositions [omits Vulgo VII, 20, 22 of Gregory]<sup>94</sup>
- VIII. 27 propositions [Ishāq's VIII, 24, 25 are not Euclidean] omits VIII, 16, 17
- IX. 38 propositions  
Order : 1-13, 20, 14-19, 21-25, 27, 26, 28-36, [with 2 new props. before 30]
- X. 109 propositions. Omits X 7, 8, 13, 16, 24, 112, 113, 114 and Vulgo X, 13 (X, 117 of Gregory);<sup>95</sup> X 31, 32 divided into 2 each order: 1-6, 9-12, 15, 14, 17-23, 26-28, 25, 29-111, 115
- XI. 41 propositions omits XI, 38 of Gregory,<sup>96</sup> XI 31, 34 divided into two each order : 1-32, 34, 33, 35-39.
- XII. 15 propositions. Omits XII, 6, 13, 14 order : 1-5, 7, 9, 8, 10, 12, 11, 15, 16-18
- XIII. 21 propositions. XIII. 1-3 each divided into 2 propositions order: 1-3, 5, 4, 6, 7, 12, 9, 10, 8, 11, 13, 15, 14, 16-18.
- XIV. [10]\* propositions (but text has 12 props.)
- XV. [6]\* propositions  
\* Heath's estimates  
478

Note : This material follows Heath's Euclid I, 79-83, and Klamroth (62).

<sup>94</sup> Vulgo VII, 20 (Heiberg's Euclid, vol. II, 428-31): "Si tres numeri proportionales sunt, productum extremorum aequale est quadrato medii et si productum extremorum aequale est quadrato medii, tres numerii illi proportionales sunt." "Vulgo VII, 22 (Heiberg's Euclid, vol. II, 431-33) : "Si tres numeri dati sunt et alii iis multitudine aequales, duo simul coniuncti et in eadem ratione, et proportio eorum perturbata est, etiam ex aequo in eadem ratione erunt".

<sup>95</sup> Authors were unable to check this reference.

<sup>96</sup> Vulgo XI, 38 (Heiberg's Euclid, vol. IV, 354-355): "Si planum ad planum perpendiculare est, et a puncto aliquo alterius plani ad alterum planum perpendicularis ducitur, perpendicularis ducta in communem planorum sectionem cadet".

**Appendix I (C).**

al-Hajjāj Elements

(Codex Leidensis 399, I)

I. 47 propositions missing I, 45

(1 ————— 44, 46-47)

II. 14 propositions

III. 36 propositions

IV. 16 propositions

V. 25 propositions

(1-11, 13, 12, 14-25)

VI. 33 propositions

(1-8, 13, 11, 12, 9, 10, 14-17, 19, 20, 18, 21-22, 24, 26, 23,  
25, 27-30, 32, 31, 33)

[See (11) for further details]

## Appendix II

Elements V, 7, 15 from al-Shifā' (Sabra's edition 160-161, 166-167)

(٧)

ا مثل ب، فنسبتها إلى ح واحدة، ونسبة ح إليها واحدة.

$$\frac{\frac{ه}{ب}}{\frac{د}{ا}} = \frac{ز}{ح}$$

رسم رقم ١٣٤

فأخذ د؛ ه أضعافاً متساوية لها، وز ل ح كيف ما اتفق.  
 ف ه مثل ه، فنقصانها وزيادتها ومساواتها ل ز واحدة، وهما أضعاف  
 متساوية للأول والثالث، فنسبة ا، ب إلى ح واحدة، وكذلك نسبة ح إليها واحدة،  
 وبالعكس إذا كانت النسب واحدة فهي متساوية.

(١٥)

ا ب فيه من ح، ما في د ه من ز، فنسبة ا ب إلى د ه ك ح إلى ز.  
 ونقسم ا ب ب ع، ط على ح. د ه ب ل، م على ز.

$$\frac{\frac{ا ب ب ع}{ب} \frac{ط على ح}{ح}}{\frac{د ه ب ل}{ب} \frac{م على ز}{ز}} = \frac{ا ح}{ب ط} = \frac{د ل}{م ز}$$

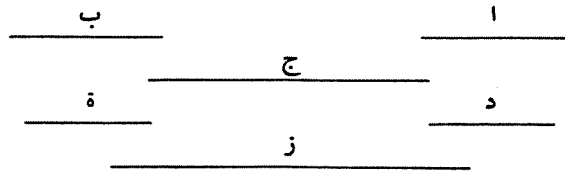
رسم رقم ١٤١

أعني ا ب، الى التوالي كلها، أعني د ه ك ا ع إلى د ل أعني ح، ز

Ishāq. (K.)

الشكل زه

الاقدار المتساوية نسبتها الى قدر آخر واحدة ونسبته هو أيضا اليها واحدة .  
 مثاله ان قدرى ا ب متساويان وقدر ج قدر آخر فاقول ان [نسبة ا الى ج  
 ك] نسبة ب الى ج ونسبة ج أيضا الى ا كنسبته الى ب . بحيث ذلك اننا نأخذ لقدرى  
 ا ب اضعافا متساوية وهي دة ولقدر ج اضعافا ما كانت وهي ز واضعاف د لقدر  
 ا كاضعاف ة لقدر ب وا مثل ب قد مثل ة وقدر آخر ما كان فقدر ا دة ان  
 مساويان معا لقدر ز واما زائدان معا عليه واما ناقصان معا عنه وهما اضعاف متساوية  
 لقدرى ا ب وز اضعاف لقدر ج فنسبة ا الى ج كنسبة ب الى ج . واقول ان نسبة  
 ج الى ا نسبته الى ب لان تديرهم واحد وقد تبين ان دة متساويان وان ز اما  
 مساويها مع [واما زائد عليها معا] واما نقص عنها معا وز اضعاف لقدر ج ودة  
 اضعاف متساوية لقدرى ا ب فنسبة ج الى ا كنسبته الى ب وذلك ما أردنا ان نبين .



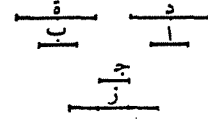
الشكل يه

الاجزاء التي اضعافها متساوية فان نسبة بعضها الى بعض كنسبة اضعافها  
 بعضها الى بعض .  
 مثاله ان اضعاف ا ب لقدر ج كاضعاف دة لقدر ز فاقول ان نسبة ا ب  
 الى دة كنسبة ج الى ز .  
 برهانه اننا نقسم ا ب بقدر ج فاقسامه ا ح حط طب وهي متساوية ونقسم دة  
 بقدر ز فاقسامه د ل لم مه وهي متساوية وعدة ا ح حط طب بعدة (؟) [كعدة  
 vielleicht] د ل لم مه فنسبة ا ح الى د ل كنسبة حط الى لم وكنسبة طب الى مه  
 ونسبة الواحد الى قرينة كنسبة الجميع الى الجميع فنسبة ا ح الى د ل كنسبة ا ب الى  
 دة واح مثل ج ودل مثل ز فنسبة ا ب الى دة كنسبة ج الى ز وذلك ما أردنا  
 ان نبين .



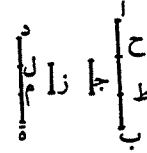
## الشكل السابع من المقالة الخامسة

المقادير المتساوية نسبتها الي مقدار آخر نسبة واحدة واذا نسب إليها ذلك المقدار فان نسبتة أيضا إليها واحدة مثاله ان مقداري  $\bar{a}$   $\bar{b}$  متساوين ومقدار  $\bar{c}$  مقدار آخر فاقول ان نسبة  $\bar{a}$  الي  $\bar{c}$  كنسبة  $\bar{b}$  الي  $\bar{c}$  ونسبة  $\bar{c}$  أيضا الي  $\bar{a}$  كنسبته الي  $\bar{b}$  برهانه انا ناخذ لمقداري  $\bar{a}$   $\bar{b}$  اضعافا متساوية ونزل عنها مقدارا  $\bar{d}$  وناخذ لمقدار  $\bar{c}$  اضعافا آخر أي الاضعاف كانت ونزل عنها مقدار زفن اجل ان ما في  $\bar{d}$  من اضعاف  $\bar{a}$  فرض مساويا لما في  $\bar{e}$  من اضعاف  $\bar{b}$  و  $\bar{a}$  فرض مساويا لمقدار  $\bar{b}$  واذا كانت مقادير متساوية واخذت لها اضعاف متساوية فان المقادير التي هي اضعاف المقادير كمتساوية هي ايضا متساوية فمقدار  $\bar{d}$  مساو لمقدار  $\bar{e}$  ومقدار آخر مفروض وهو مقدار  $\bar{z}$  فظاهر ان مقدار  $\bar{d}$  ان كان زائدا علي مقدار  $\bar{r}$  فان مقدار  $\bar{e}$  ايضا زائد علي  $\bar{z}$  وان كان  $\bar{d}$  ناقصا عن  $\bar{z}$  فان  $\bar{e}$  ناقص عن  $\bar{z}$  وان كان مساويا له فهو مساو له فقدر  $\bar{d}$  اما ان يكونا زائدين معا علي  $\bar{r}$  واما ناقصين معا عنه واما مساوين معا له وهما اضعاف متساوية لمقدري  $\bar{a}$   $\bar{b}$  ومقدار  $\bar{z}$  اضعاف لمقدار  $\bar{c}$  فنسبة  $\bar{a}$  الي  $\bar{c}$  كنسبة  $\bar{b}$  الي  $\bar{c}$  واقول ايضا ان نسبة  $\bar{c}$  الي  $\bar{a}$  كنسبته الي  $\bar{b}$  ونبين ايضا كما بينا ان  $\bar{d}$   $\bar{e}$  متساويان وان  $\bar{z}$  ان كان زائدا علي  $\bar{d}$  فهو ايضا زائد علي  $\bar{e}$  وان كان ناقصا عن  $\bar{d}$  فانه ايضا ناقص عن  $\bar{e}$  وان كان مساويا له فهو ايضا مساو له فقدر  $\bar{z}$  اما ان يكون زائدا علي  $\bar{d}$   $\bar{e}$  معا واما ناقصا منها معا واما مساويا لهما معا ومقدار  $\bar{z}$  فرض اضعافا لمقدار  $\bar{c}$  ومقدارا  $\bar{d}$   $\bar{e}$  فرضا اضعافا متساوية لمقدري  $\bar{a}$   $\bar{b}$  فنسبة  $\bar{c}$  اذن الي  $\bar{a}$  كنسبته الي  $\bar{b}$  وذلك ما اردنا ان نبين.



الشكل الخامس عشر من المقالة الخامسة

إذا كانت مقاديرُ واخذها اضعاف متساوية فان نسبة المقادير بعضها  
 الي بعض كنسبة اضعافها بعضها الي بعض مثالة ان مقداري ج ز مفروضان  
 وقد أخذلها اضعاف متساوية وهي ا ب ود ه وفي ا ب من اضعاف ج مثل  
 ما في د ه من اضعاف ز فاقول ان نسبة ج الي ز كنسبة ا ب الي د ه برهانه انا  
 نفصل ما في ا ب ود ه من اضعاف ج ز وهي  
 ا ح ط ب ود ل م ه فقادير ا ح ط ب متساوية لان كل  
 واحد منها مثل ج وايضا مقادير د ل م ه متساوية لان كل واحد منها مثل  
 ز وعدة ا ح و ح ط و ط ب مساوية لعدة د ل و ل م وم ه فظاهر ان نسبة  
 ا ح الي د ل كنسبة ح ط الي ل م وكنسبة ط ب الي م ه فبين من برهان ١٣  
 من ه ان نسبة ا ح الي د ل كنسبة ا ب الي د ه واح مساو لمقدار ج ود ل  
 مساو لمقدار ز فنسبة ج الي ز كنسبة ا ب الي د ه وذلك ما أردنا ان نبين .:



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